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## SCALE COVARIANCE AND $G$ -VARYING COSMOLOGY. III. THE $(m, z)$ , $(\theta_m, z)$ , $(\theta_i, z)$ , AND $[N(m), m]$ TESTS

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### ABSTRACT

Within the context of the recently proposed scale-covariant cosmology we present in this paper: (a) the full solution to Einstein gravitational equations in atomic units for a matter-dominated universe, (b) the study of the magnitude versus redshift relation for elliptical galaxies, (c) the derivation of the evolutionary parameter used in (b), (d) the isophotal angular diameters versus redshifts, (e) the metric angular diameters versus redshifts, and finally (f) the  $N(m)$  versus magnitude relation for QSOs and their  $m$  versus  $z$  relation.

The results, both in graphical and tabular form, are presented for the four gauges [i.e., relation between  $G$  and the scale function  $\beta(t)$ ] introduced and studied in previous works.

*No contradiction between the new theory and the data is found with any of the tests studied.*

If we chose the gauges with  $\epsilon < 0$ ,  $\beta \sim t^{-\epsilon}$ , as suggested by a recent analysis of the time variation, of the Moon's period, only an open universe can fit the data.

For the gauges with  $\epsilon > 0$ , the results become very similar to those of standard cosmology.

As for QSOs, since we cannot evaluate the evolutionary parameter as we did in the case of elliptical galaxies, we determine it by requiring that the  $\log N(m)$  versus  $m$  relation has an average slope derived from recently published data. It is shown that such an evolutionary parameter yields a satisfactory  $m$  versus  $z$  relation for QSOs. When the same procedure is applied to standard cosmology, the results are less satisfactory.

Considerations concerning the radiation-dominated universe are briefly discussed at the end. Finally, we introduce the concept of "lens effect" (§ VIII). The time dependence of  $G$  can be thought of as indicating that gravitational phenomena, when viewed through atomic measurements, can be either magnified or reduced, thus changing the relation between density of matter and space curvature. The implications of this new feature for the geometry of the universe are discussed.

*Subject headings:* cosmology — galaxies: redshifts — gravitation — quasars — relativity

### I. INTRODUCTION

In previous papers (Canuto *et al.* 1977; Canuto and Hsieh 1978b; hereafter Papers I and II, respectively) the theoretical structure of scale-covariant gravitation was introduced so that gravitational phenomena can be studied in atomic units, assuming quite generally that the gravitational and atomic *dynamical units* are related through the scaling function  $\beta(x)$ . It was emphasized that Einstein's theory of gravitation remains unaltered if one limits oneself to using gravitational clocks. However, when observations are made and recorded with atomic instruments, one goes beyond the realm of purely gravitational dynamics. The function  $\beta(x)$  was thus explicitly introduced to parametrize our ignorance of the correct coupling of gravitation to atomic dynamics over large cosmological scales. It was argued that any variation of the gravitational constant  $G$  must be interpreted as a relative variation of gravitational and atomic dynamics. Furthermore, such a variation does not contradict the improved experimental confirmation of Einstein's general theory of relativity.

Along with the new formalism, we derived *new conservation laws which must be adhered to in place of the standard conservation laws*. In fact, the inconsistency of studying the effects of varying  $G$  with the imposition of the standard conservation laws has been repeatedly pointed out (see, e.g., Canuto, Hsieh, and Owen 1978). A careful examination of the thermodynamic laws consistent with the scale-covariant gravitation was given in Paper II, where a semi-classical description of particles and photons in terms of the geometrical parameters of the scale-covariant theory can be found. It was furthermore demonstrated in the above paper that a consistent analysis of the radiation problem allows one to interpret the observed background radiation as the remnant of an equilibrium radiation of an earlier epoch, just as in standard cosmology (see also Canuto and Hsieh 1978a).

Papers I and II will serve as a foundation upon which the astrophysical analysis will be based.

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In the present paper we consider well-known cosmological observations and see if they are in agreement with the predictions of the scale-covariant theory. Before entering into a detailed discussion, we indicate here some qualitative features peculiar to the scale-covariant framework, normalizing our description to the well-known standard cosmology.

First, it must be noted that the fundamental cosmological observations, such as redshift and luminosity measurements, are of atomic nature. To deduce from these measurements the geometrical structure of the universe as determined by gravitation, *one must first disentangle the relation between atomic and gravitational dynamics*. That is, one must know the scaling function  $\beta(t)$  (see Canuto, Hsieh, and Owen 1978 for a thorough discussion of observational and theoretical determinations of the time derivatives of  $\beta$ ). Specifically, if the metric (in atomic units) describing the cosmological spacetime has the Robertson-Walker form, then the time variation of the scale factor  $R(t)$ , which is governed by equation (2.1) below, depends not only on gravitational dynamics but also on the time evolution of the function  $\beta$ . Thus, only if  $\beta(t)$  is known can one deduce the true geometrical structure of the universe. The latter is still characterized by  $\bar{H}_0$  and  $\bar{q}_0$ , *defined in the usual manner in terms of the gravitational metric*. Clearly, it is  $\bar{H}_0$ ,  $\bar{q}_0$ , and not  $H_0$ ,  $q_0$ , defined in terms of the atomic metric, which are directly related to the purely gravitational parameters such as the curvature of space  $k$ , see equation (2.3).

We note that with a prescribed  $\beta$ , the cosmological equations (2.1) become a proper set of second-order differential equations so that the scale-covariant theory can be subjected to cosmological tests just as general relativity is tested with standard cosmology. The objectives are also similar: first, one must demonstrate the compatibility of the observed data with the predictions of the theoretical model. (Other attempts to prove the consistency of  $G \sim t^{-1}$  with the  $m$  versus  $z$  relation have notoriously failed. See the discussion.) If that is achieved, one then proceeds to determine the parameters which characterize the model, namely  $\bar{H}_0$  and  $\bar{q}_0$ . The functional forms of  $\beta$  considered have been derived from various impositions of gauge conditions. Hence the cosmological tests must be considered as tests of the specific gauge conditions.

It is natural at this point to ask whether cosmological observations can be used as a constraint on the scaling function  $\beta(t)$  and thus constitute an alternative observational determination of  $\beta$ . We can anticipate the results: for all scaling functions of the forms given by (4.1), there exist theoretically allowed parameters which make the theoretical predictions compatible with observation. [Whether we should prefer an increasing or decreasing  $\beta(t)$  must be based on purely gravitational tests as discussed in § XIV.]

A common feature that plagues all cosmological investigations is the evolutionary correction, i.e., the intrinsic variation of the sources. As is well known, the inclusion of evolutionary correction has caused difficulties over the interpretation of the cosmological data, for the truly geometrical effects are not easily separated from the evolutionary effects. To a large extent, these uncertainties persist in the scale-covariant framework. Even though the evolutionary effects can be estimated, as we have done in Appendix A, the extent to which the various evolutionary processes are operative is not known. Hence, the theoretical estimates of Appendix A should be used as a guide to the evolutionary effects, and for this reason, in the tables and graphs below, we give results for different values of the evolutionary index  $e$  defined in equation (7.8).

The use of QSOs as cosmological probes has been another controversial subject in standard cosmology. Treating QSOs as objects at distances indicated by their redshifts, we use the observed slope of the  $N$ - $m$  relation to determine the evolutionary index for these objects. When such an evolution is used to evaluate the magnitude-redshift relation, we obtain theoretical  $m$ - $z$  curves which fit the data points representing the QSOs. When we apply a similar procedure using standard cosmology, the fitting of the data is less successful.

Having thus given a preview of the highlights of the present paper as a general orientation, we proceed with the details of the analysis by first solving the cosmological equations for each choice of the scaling function. Luminosity and angular diameters are then derived as functions of the observed redshift. Except when analytic solutions can be obtained, results are presented in tabular form. Comparisons with existing data are then made and presented in figures.

## II. COSMOLOGY

As shown in previous papers, Einstein's equations in atomic units and for a Robertson-Walker metric read (I, eq. [3.2]: in this paper we shall take  $\Lambda = 0$ )

$$\left(\frac{\dot{R}}{R} + \frac{\dot{\beta}}{\beta}\right)^2 + \frac{k}{R^2} = \frac{8\pi}{3} G\rho, \quad (2.1a)$$

$$\frac{\ddot{R}}{R} + \frac{\ddot{\beta}}{\beta} + \frac{\dot{\beta}\dot{R}}{\beta R} - \frac{\dot{\beta}^2}{\beta^2} = -\frac{4\pi G}{3}(3p + \rho), \quad (2.1b)$$

where the derivatives are taken with respect to atomic time,  $t$ .  $R(t)$  is the scale factor and  $p$ ,  $\rho$ , and  $G$  are in general functions of the scale function  $\beta$  which depends on the chosen gauge. If one decides to work in the so-called Einstein gauge,  $\beta = 1$ , then the atomic and Einstein times,  $\beta dt = d\bar{t}$ , coincide. The dots in equations (2.1) would then represent derivatives with respect to  $\bar{t}$ . Clearly, if  $\beta = 1$ , all the  $\beta$  terms disappear and equations (2.1) reduce to the familiar Einstein equations. It is the contention of our theory that the assumption  $\beta = \text{const.}$ , usually made

in cosmology, and leading to an elimination of the difference between Einstein and atomic times (except for a time-independent large numerical factor) is unproven and should be tested by using observational data.

For that reason, we have extended the theory so as to incorporate a nonconstant  $\beta$ . The extra terms in equations (2.1) represent precisely this contribution. Should it turn out that the data can be fitted only with  $\beta = \text{const.}$ , we would have legitimized this gauge choice, heretofore made *a priori*. In this sense our theory does not change the physics underlying Einstein theory, nor does it replace the Einstein equations with new ones. Rather, it investigates the circumstances under which the assumption of a constant  $\beta$  is actually verified, if at all. The scale over which  $\beta$  varies is the age of the universe; and therefore, the effect of  $\beta$  can be significant only if one extends the analysis to high-redshift objects so that a good portion of the universe is actually investigated. From now on, quantities referring to Einstein units will be denoted by a bar.

With  $\bar{R}(t) = \beta R(t)$ ,  $d\bar{t} = \beta(t)dt$ , it is easy to derive the following relations ( $\beta_0 = 1$ ):

$$\bar{H}_0 = H_0 + h_0; \quad h_0 \equiv (\dot{\beta}/\beta)_0, \quad Q_0 \equiv -\ddot{\beta}(\beta/\dot{\beta}^2)_0, \quad (2.2)$$

$$\frac{k}{\bar{R}_0^2} = (2\bar{q}_0 - 1)\bar{H}_0^2 = (\rho_0/\rho_c - \bar{H}_0^2/H_0^2)H_0^2, \quad (2.3)$$

$$\frac{\rho_0}{\rho_c} = 2\bar{q}_0(\bar{H}_0/H_0)^2, \quad (2.4)$$

$$\bar{q}_0 = q_0(H_0/\bar{H}_0)^2 + (1 + Q_0)(h_0/\bar{H}_0)^2 - (h_0H_0)/\bar{H}_0^2, \quad (2.5)$$

where

$$\bar{q}_0 = -\left(\frac{\bar{R}''\bar{R}}{\bar{R}'^2}\right)_0, \quad q_0 = -\left(\frac{\ddot{R}R}{\dot{R}^2}\right)_0,$$

and a prime denotes  $d/d\bar{t}$ ; a dot,  $d/dt$ . From Einstein's equations (2.1), one can also derive the following energy conservation law (I, eq. [2.38]):

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) = -\rho \frac{d(G\beta)/dt}{G\beta} - 3p\frac{\dot{\beta}}{\beta}. \quad (2.6)$$

For an equation of state of the form

$$p = c_s^2\rho$$

equation (2.6) can be integrated exactly with the result (see I, eq. [3.3])

$$\rho = \rho_0 \left(\frac{R_0}{R}\right)^{3(1+c_s^2)} \frac{G_0}{G(\beta)} \frac{1}{\beta^{1+3c_s^2}}. \quad (2.7)$$

We now have all the necessary ingredients to study the behavior of the scale factor  $R(t)$  versus  $t$ .

### a) World Models

Substituting equation (2.7) in (2.1), and integrating, we obtain ( $c_s^2 = 0$ )

1)  $k = 0$ :

$$\begin{aligned} \beta(t) \frac{R(t)}{R_0} &= \left[ \frac{3}{2}\bar{H}_0 \int_0^t \beta(t)dt \right]^{2/3}; \\ \frac{8\pi}{3} G_0 \rho_0 &= \frac{k}{R_0^2} + \bar{H}_0^2 \equiv A = B + \bar{H}_0^2. \end{aligned} \quad (2.8)$$

2)  $k = -1$ :

$$\begin{aligned} \beta(t) \frac{R(t)}{R_0} &= \frac{A}{2B} (\cosh \psi - 1); \quad \int_0^t \beta(t)dt = (A/2B^{3/2})(\sinh \psi - \psi); \\ A/2B &= \bar{q}_0(1 - 2\bar{q}_0)^{-1}; \quad A/2B^{3/2} = \bar{q}_0/\bar{H}_0(1 - 2\bar{q}_0)^{3/2}. \end{aligned} \quad (2.9)$$

3)  $k = +1$ :

$$\begin{aligned} \beta(t) \frac{R(t)}{R_0} &= \frac{A}{2B} (1 - \cos \theta); \quad \int_0^t \beta(t)dt = (A/2B^{3/2})(\theta - \sin \theta); \\ A/2B &= \bar{q}_0(2\bar{q}_0 - 1)^{-1}; \quad A/2B^{3/2} = \bar{q}_0/\bar{H}_0(2\bar{q}_0 - 1)^{3/2}. \end{aligned} \quad (2.10)$$

Numerical solutions for  $R(t)$  versus  $t$  are presented in Tables 1–4 for different values of  $\bar{q}_0$ .

b) *Radiation-dominated Universe:  $c_s^2 = \frac{1}{3}$*

1)  $k = 0$ :

$$\beta(t)R(t)/R_0 = \left[ 2\bar{H}_0 \int_0^t \beta(t)dt \right]^{1/2}. \quad (2.11)$$

2)  $k = -1$ :

$$\beta(t) \frac{R(t)}{R_0} = \left( \frac{2\bar{q}_0}{1 - 2\bar{q}_0} \right)^{1/2} \left\{ \left[ 1 + \frac{1 - 2\bar{q}_0}{(2\bar{q}_0)^{1/2}} \bar{H}_0 \int_0^t \beta(t)dt \right]^2 - 1 \right\}^{1/2}. \quad (2.12)$$

3)  $k = +1$ :

$$\beta(t) \frac{R(t)}{R_0} = \left( \frac{2\bar{q}_0}{2\bar{q}_0 - 1} \right)^{1/2} \left\{ \left[ 1 + \frac{2\bar{q}_0 - 1}{(2\bar{q}_0)^{1/2}} \bar{H}_0 \int_0^t \beta(t)dt \right]^2 - 1 \right\}^{1/2}. \quad (2.13)$$

c) *The Age of the Matter-dominated Era,  $t_0$*

This age can easily be determined by putting  $t = t_0$ ,  $R = R_0$  ( $\beta_0 = 1$ ), in the upper limit of the integrals (2.8)–(2.10), thus getting

$k = 0$ :

$$t_0 H_0 = \frac{2}{3} \left[ \left( 1 + h_0/H_0 \right) \int_0^1 \beta(x)dx \right]^{-1} \equiv \frac{2}{3} I(\beta); \quad (2.14)$$

$k = +1$ :

$$t_0 H_0 = \frac{\bar{q}_0}{(2\bar{q}_0 - 1)^{3/2}} \left[ \cos^{-1} \left( \frac{1}{\bar{q}_0} - 1 \right) - \frac{1}{\bar{q}_0} (2\bar{q}_0 - 1)^{1/2} \right] I(\beta); \quad (2.15)$$

$k = -1$ :

$$t_0 H_0 = \left[ (1 - 2\bar{q}_0)^{-1} - \bar{q}_0 (1 - 2\bar{q}_0)^{-3/2} \cosh^{-1} \left( \frac{1}{\bar{q}_0} - 1 \right) \right] I(\beta). \quad (2.16)$$

If we put  $\beta = 1$ , equations (2.14)–(2.16) reduce to the well-known expressions in Einstein units, i.e.,  $\bar{t}_0 \bar{H}_0$ . We therefore obtain in general,

$$\bar{t}_0 = t_0 \int_0^1 \beta(x)dx, \quad (2.17)$$

an expression which relates the age  $t_0$  in Einstein and atomic units. Values of  $t_0 H_0$  are presented in Tables 1–4.

### III. THE REDSHIFT

As proved in II (eq. [3.14]), the measured redshift should be defined as

$$1 + z = R_0/R, \quad (3.1)$$

i.e., as a function of the scale factor in atomic units. Within the present theory, the ratio of the scale factors in Einstein units is a convenient symbol and not a physically meaningful quantity. We shall denote it by  $\varkappa$ ,

$$1 + \varkappa = \bar{R}_0/\bar{R}. \quad (3.2)$$

However, since  $\beta R(t) = \bar{R}$ , it follows that

$$\beta(1 + \varkappa) = 1 + z. \quad (3.3)$$

We shall find it convenient to express intermediate results in terms of  $\varkappa$  instead of  $z$ . However, physical results will always be expressed in terms of  $z$ .

### IV. THE SCALE FUNCTION $\beta(t)$

As we have already discussed in great detail in the previous papers, the function  $\beta(t)$  cannot be determined from within the theory as we have formulated it. External considerations must be used which can vary depending

TABLE 1,  $\epsilon = -1$

TABLE 2,  $\epsilon = -\frac{1}{2}$ 

$\bar{q}_e$	$q_e$	$\bar{t}_e H_0$	$t_e H_0$	$\gamma$	$z$	$R/R_0$	$t/t_0$	$\beta$	$F$
0.0	0.0	1.000	1.000						
				56.140	13.835	0.067	0.067	0.260	1631.970
				25.666	7.926	0.112	0.112	0.335	355.037
				16.391	5.712	0.149	0.149	0.386	150.724
				11.903	4.501	0.182	0.182	0.426	82.745
				9.256	3.721	0.212	0.212	0.460	52.096
				7.511	3.168	0.240	0.240	0.490	35.715
				6.273	2.754	0.266	0.266	0.516	25.946
				5.349	2.429	0.292	0.292	0.540	19.656
				4.634	2.166	0.316	0.316	0.562	15.370
				4.063	1.949	0.339	0.339	0.582	12.318
				3.598	1.765	0.362	0.362	0.601	10.069
				3.211	1.607	0.384	0.384	0.619	8.364
				2.883	1.471	0.405	0.405	0.636	7.041
				2.604	1.350	0.425	0.425	0.652	5.993
				0.333	0.211	0.825	0.825	0.909	0.389
				0.250	0.160	0.862	0.862	0.928	0.281
				0.176	0.114	0.897	0.897	0.947	0.192
				0.111	0.073	0.932	0.932	0.965	0.117
				0.053	0.035	0.966	0.966	0.983	0.054
0.100	0.078	0.846	0.770						
				199.572	16.069	0.059	0.007	0.085	1510.038
				88.965	10.404	0.088	0.016	0.127	589.494
				49.481	7.514	0.117	0.028	0.169	287.729
				31.019	5.752	0.148	0.044	0.211	158.634
				20.932	4.560	0.180	0.064	0.253	94.355
				14.829	3.695	0.213	0.088	0.297	59.062
				10.863	3.036	0.248	0.116	0.340	38.322
				8.144	2.516	0.284	0.148	0.385	25.517
				6.202	2.094	0.323	0.185	0.430	17.309
				4.770	1.743	0.365	0.226	0.475	11.894
				3.687	1.447	0.409	0.273	0.522	8.240
				2.849	1.194	0.456	0.325	0.570	5.727
				2.190	0.974	0.507	0.383	0.619	3.974
				1.663	0.781	0.561	0.447	0.669	2.736
				1.238	0.612	0.620	0.519	0.720	1.853
				0.890	0.461	0.684	0.597	0.773	1.218
				0.604	0.327	0.754	0.684	0.827	0.759
				0.366	0.206	0.829	0.780	0.883	0.424
				0.167	0.098	0.911	0.885	0.941	0.180
0.500	1.000	0.667	0.500						
				67.861	7.298	0.121	0.015	0.121	121.126
				40.430	5.437	0.155	0.024	0.155	69.986
				30.157	4.582	0.179	0.032	0.179	51.150
				24.535	4.053	0.198	0.039	0.198	40.963
				20.911	3.681	0.214	0.046	0.214	34.460
				18.348	3.399	0.227	0.052	0.227	29.899
				16.423	3.174	0.240	0.057	0.240	26.498
				2.429	0.852	0.540	0.292	0.540	3.154
				2.166	0.779	0.562	0.316	0.562	2.774
				1.949	0.717	0.582	0.339	0.582	2.463
				1.765	0.663	0.601	0.362	0.601	2.204
				1.607	0.615	0.619	0.384	0.619	1.985
				1.471	0.572	0.636	0.405	0.636	1.798
				1.350	0.533	0.652	0.425	0.652	1.635
				0.256	0.121	0.892	0.796	0.892	0.271
				0.201	0.096	0.913	0.833	0.913	0.210
				0.151	0.073	0.932	0.869	0.932	0.156
				0.106	0.052	0.951	0.904	0.951	0.108
				0.065	0.032	0.969	0.939	0.969	0.066
1.000	3.241	0.571	0.356						
				103.837	8.611	0.104	0.008	0.092	103.837
				42.991	5.230	0.161	0.020	0.142	42.991
				23.109	3.616	0.217	0.037	0.191	23.109
				14.227	2.673	0.272	0.058	0.241	14.227
				9.507	2.056	0.327	0.085	0.291	9.507
				6.703	1.621	0.381	0.116	0.340	6.703
				4.903	1.300	0.435	0.152	0.390	4.903
				3.679	1.052	0.487	0.192	0.439	3.679
				2.810	0.657	0.539	0.238	0.487	2.810
				2.170	0.699	0.589	0.287	0.536	2.170
				1.686	0.569	0.637	0.341	0.584	1.686
				1.311	0.460	0.685	0.399	0.632	1.311
				1.015	0.369	0.730	0.462	0.680	1.015
				0.777	0.291	0.775	0.528	0.727	0.777
				0.583	0.224	0.817	0.598	0.773	0.583
				0.423	0.166	0.858	0.672	0.820	0.423
				0.289	0.116	0.896	0.749	0.866	0.289
				0.177	0.072	0.933	0.830	0.911	0.177
				0.081	0.034	0.968	0.913	0.956	0.081

TABLE 3,  $\epsilon = \frac{1}{2}$ 

$\bar{q}_e$	$q_e$	$\bar{t}_e H_0$	$t_e H_0$	$\beta$	$z$	$R/R_e$	$t/t_e$	$\beta$	$F$
0.0	0.0	1.000	1.000						
				0.965	2.863	0.259	0.259	1.965	1.431
				0.928	2.715	0.269	0.269	1.928	1.358
				0.891	2.576	0.280	0.280	0.891	1.288
				0.856	2.445	0.290	0.290	0.856	1.222
				0.822	2.320	0.301	0.301	0.822	1.160
				0.790	2.203	0.312	0.312	0.790	1.101
				0.758	2.091	0.324	0.324	0.758	1.045
				0.728	1.985	0.335	0.335	0.728	0.993
				0.698	1.885	0.347	0.347	0.698	0.942
				0.670	1.789	0.359	0.359	0.670	0.895
				0.643	1.698	0.371	0.371	0.643	0.849
				0.616	1.612	0.383	0.383	0.616	0.806
				0.590	1.529	0.395	0.395	0.590	0.765
				0.565	1.451	0.408	0.408	0.565	0.725
				0.143	0.306	0.766	0.766	1.143	0.153
				0.111	0.235	0.810	0.810	1.111	0.117
				0.081	0.169	0.856	0.856	0.081	0.084
				0.053	0.108	0.903	0.903	0.053	0.054
				0.026	0.052	0.951	0.951	0.026	0.026
0.100	0.066	0.846	0.923						
				1.860	9.478	0.095	0.075	3.663	3.181
				1.693	8.131	0.110	0.087	3.390	2.843
				1.538	6.974	0.125	0.101	3.142	2.465
				1.393	5.976	0.143	0.118	2.915	2.163
				1.258	5.115	0.164	0.136	2.708	1.843
				1.132	4.369	0.186	0.158	2.518	1.651
				1.014	3.721	0.212	0.182	2.344	1.435
				0.904	3.159	0.240	0.210	2.185	1.241
				0.800	2.669	0.273	0.241	2.038	1.067
				0.703	2.241	0.309	0.276	1.903	0.911
				0.612	1.867	0.349	0.316	1.778	0.771
				0.527	1.539	0.394	0.362	1.663	0.645
				0.446	1.251	0.444	0.413	1.557	0.532
				0.371	0.999	0.500	0.470	1.458	0.430
				0.299	0.776	0.563	0.535	1.307	0.339
				0.232	0.581	0.633	0.608	1.283	0.250
				0.169	0.408	0.710	0.690	1.204	0.182
				0.109	0.255	0.797	0.782	1.131	0.115
				0.053	0.120	0.893	0.885	1.063	0.054
0.500	0.200	0.667	0.833						
				0.585	2.164	0.316	0.251	1.996	0.652
				0.583	2.154	0.317	0.252	1.992	0.650
				0.581	2.143	0.318	0.253	1.988	0.647
				0.579	2.133	0.319	0.254	1.984	0.645
				0.577	2.123	0.320	0.255	1.980	0.642
				0.575	2.112	0.321	0.256	1.976	0.640
				0.573	2.102	0.322	0.257	1.972	0.637
				0.440	1.488	0.402	0.335	1.728	0.480
				0.424	1.418	0.414	0.347	1.698	0.461
				0.408	1.351	0.420	0.359	1.670	0.442
				0.392	1.287	0.437	0.371	1.643	0.425
				0.377	1.226	0.449	0.383	1.616	0.407
				0.362	1.167	0.462	0.395	1.590	0.390
				0.348	1.111	0.474	0.408	1.565	0.374
				0.110	0.298	0.770	0.731	1.170	0.113
				0.089	0.237	0.808	0.774	1.136	0.091
				0.069	0.181	0.847	0.819	1.105	0.070
				0.050	0.129	0.886	0.865	1.075	0.050
				0.031	0.080	0.926	0.912	1.047	0.031
1.000	0.306	0.571	0.785						
				0.897	4.491	0.182	0.119	2.895	0.897
				0.817	3.914	0.204	0.137	2.703	0.817
				0.744	3.410	0.227	0.156	2.529	0.744
				0.675	2.969	0.252	0.178	2.370	0.675
				0.610	2.582	0.279	0.202	2.224	0.610
				0.550	2.241	0.309	0.229	2.090	0.550
				0.494	1.939	0.340	0.258	1.968	0.494
				0.441	1.673	0.374	0.291	1.855	0.441
				0.391	1.436	0.411	0.326	1.751	0.391
				0.345	1.225	0.450	0.365	1.654	0.345
				0.301	1.036	0.491	0.408	1.565	0.301
				0.259	0.868	0.535	0.455	1.483	0.259
				0.220	0.716	0.583	0.505	1.407	0.220
				0.183	0.580	0.633	0.501	1.330	0.183
				0.148	0.458	0.686	0.620	1.270	0.148
				0.115	0.348	0.742	0.685	1.208	0.115
				0.084	0.248	0.802	0.755	1.151	0.084
				0.055	0.157	0.864	0.831	1.097	0.055
				0.027	0.075	0.930	0.912	1.047	0.027

TABLE 4,  $\epsilon = 0.92$ 

$\bar{q}_e$	$q_e$	$\bar{t}_e H_0$	$t_e H_0$	$\beta$	$z$	$R/R_0$	$t/t_0$	$\beta$	$F$
0.0	0.0	1.000	1.000						
					0.149	4.676	0.176	0.176	4.940
					0.146	4.466	0.183	0.183	4.771
					0.142	4.263	0.190	0.190	4.608
					0.139	4.069	0.197	0.197	4.451
					0.135	3.882	0.205	0.205	4.300
					0.132	3.703	0.213	0.213	4.155
					0.128	3.531	0.221	0.221	4.015
					0.125	3.365	0.229	0.229	3.880
					0.122	3.206	0.238	0.238	3.750
					0.118	3.054	0.247	0.247	3.624
					0.115	2.907	0.256	0.256	3.503
					0.112	2.766	0.266	0.266	3.387
					0.109	2.631	0.275	0.275	3.275
					0.105	2.500	0.286	0.286	3.167
					0.034	0.521	0.657	1.471	0.035
					0.027	0.397	0.716	0.716	1.360
					0.020	0.284	0.779	0.779	1.259
					0.013	0.181	0.847	0.847	1.165
					0.007	0.086	0.921	0.921	1.079
0.100	0.012	0.846	0.988		0.243	23.379	0.041	0.039	19.621
					0.228	19.539	0.049	0.047	16.722
					0.214	16.311	0.058	0.056	14.257
					0.200	13.595	0.069	0.066	12.160
					0.186	11.310	0.081	0.079	10.376
					0.173	9.387	0.096	0.093	8.856
					0.160	7.768	0.114	0.111	7.561
					0.146	6.403	0.135	0.132	6.459
					0.133	5.254	0.160	0.156	5.518
					0.120	4.285	0.189	0.185	4.717
					0.108	3.467	0.224	0.220	4.033
					0.095	2.777	0.265	0.260	3.450
					0.083	2.195	0.313	0.308	2.952
					0.070	1.704	0.370	0.365	2.526
					0.058	1.289	0.437	0.432	2.163
					0.046	0.938	0.516	0.512	1.852
					0.035	0.642	0.609	0.605	1.587
					0.023	0.391	0.719	0.716	1.360
					0.011	0.179	0.848	0.846	1.166
0.500	0.027	0.667	0.973		0.084	3.324	0.231	0.222	3.991
					0.083	3.311	0.232	0.223	3.979
					0.083	3.297	0.233	0.224	3.967
					0.083	3.284	0.233	0.224	3.956
					0.083	3.271	0.234	0.225	3.944
					0.083	3.257	0.235	0.226	3.932
					0.082	3.244	0.236	0.226	3.921
					0.069	2.401	0.294	0.284	3.180
					0.068	2.299	0.303	0.293	3.090
					0.066	2.200	0.313	0.303	3.002
					0.064	2.104	0.322	0.312	2.917
					0.062	2.011	0.332	0.322	2.835
					0.061	1.922	0.342	0.332	2.755
					0.059	1.835	0.353	0.343	2.677
					0.023	0.503	0.665	0.658	1.470
					0.019	0.400	0.714	0.708	1.374
					0.015	0.304	0.767	0.761	1.285
					0.011	0.215	0.823	0.818	1.202
					0.007	0.133	0.882	0.879	1.125
1.000	0.036	0.571	0.966		0.135	13.088	0.071	0.065	12.410
					0.127	11.180	0.082	0.075	10.805
					0.119	9.538	0.095	0.087	9.414
					0.112	8.124	0.110	0.101	8.208
					0.104	6.905	0.126	0.118	7.162
					0.096	5.855	0.146	0.136	6.252
					0.089	4.948	0.168	0.158	5.462
					0.081	4.164	0.194	0.183	4.775
					0.074	3.487	0.223	0.211	4.177
					0.067	2.902	0.256	0.244	3.657
					0.060	2.395	0.295	0.282	3.203
					0.053	1.956	0.338	0.326	2.807
					0.046	1.575	0.388	0.376	2.462
					0.039	1.245	0.445	0.433	2.161
					0.032	0.959	0.510	0.498	1.897
					0.026	0.710	0.585	0.574	1.667
					0.019	0.494	0.669	0.660	1.466
					0.013	0.306	0.766	0.758	1.290
					0.006	0.142	0.875	0.871	1.135

NOTE.—The value  $\epsilon = 0.92$  is derived by requiring that the cosmological constant  $\Lambda$ , computed from gauge theory at early times and which scales like  $\Lambda_0 \beta^2$ , be compatible with today's value.

on which data are deemed most significant. In the previous papers we discussed different functional dependences of  $\beta$  on  $t$ , represented by a single function

$$\beta(t) = (t_0/t)^\epsilon, \quad \epsilon = \mp 1; \mp \frac{1}{2}, \quad (4.1)$$

corresponding to the following gauges:

$$G \sim \beta^{-1}, \beta, \beta^{-2}, \beta^2. \quad (4.2)$$

If  $G \sim t^{-1}$ , the first two gauges are the ones proposed by Dirac in 1938, 1973, and 1979, respectively. The third one was proposed by Canuto and Hsieh (1978a).

## V. A COMPLETE SOLUTION

### a) General Relations

Using (2.17), (4.1), and (2.14)–(2.16), we now obtain

$$t_0 = \bar{t}_0(1 - \epsilon), \quad H_0 t_0 = (1 - \epsilon)\bar{H}_0 \bar{t}_0 + \epsilon. \quad (5.1)$$

Since it is easier to evaluate  $\bar{H}_0 \bar{t}_0$  than  $H_0 t_0$ , we shall use (5.1) as the equation defining  $H_0 t_0$ . The relation (2.5) between  $\bar{q}_0$  and  $q_0$  can also be considerably simplified. It becomes

$$\begin{aligned} \frac{k}{R_0^2} &= \left[ \frac{\rho_0}{\rho_c} - \left( 1 - \frac{\epsilon}{H_0 t_0} \right)^2 \right] H_0^2 = (2\bar{q}_0 - 1)\bar{H}_0^2, \\ \frac{\rho_0}{\rho_c} &= 2\bar{q}_0 + \frac{2\epsilon}{H_0 t_0} \left[ 1 - \frac{1}{H_0 t_0} \right], \\ q_0 &= \bar{q}_0 \left( \frac{\bar{H}_0}{H_0} \right)^2 + \frac{\epsilon}{(H_0 t_0)^2} - \frac{\epsilon}{H_0 t_0}. \end{aligned} \quad (5.2)$$

### b) The Analytic Case: $k = 0$ , $\bar{q}_0 = \frac{1}{2}$

The case  $k = 0$  can be treated analytically, and we shall therefore present the results corresponding to it before giving the full numerical solution. We present only the zero-pressure case, since these results are the ones of physical interest for the purposes of the present paper.

From equation (2.8) we obtain

$$R(t) = R_0(t/t_0)^r, \quad r \equiv \frac{1}{3}(2 + \epsilon). \quad (5.3)$$

Since  $\bar{H}_0 \bar{t}_0$  is equal to  $\frac{2}{3}$  (see eq. [2.17] with  $\beta = 1$ ), it follows from (5.1) or (5.3) that

$$H_0 t_0 = \frac{1}{3}(2 + \epsilon), \quad \bar{H}_0 = H_0 \frac{2(1 - \epsilon)}{2 + \epsilon}. \quad (5.4)$$

From (5.2) we finally obtain

$$q_0 = \frac{1 - \epsilon}{2 + \epsilon} \quad (5.5)$$

or

$$H_0 t_0 q_0 = \frac{1}{3}(1 - \epsilon).$$

At the same time, from (3.3) we derive

$$1 + z = (1 + z)^{(2-2\epsilon)/(2+\epsilon)}, \quad \beta(t) = (1 + z)^{3\epsilon/(2+\epsilon)}. \quad (5.6)$$

Explicitly, for  $k = 0$ ,  $\bar{q}_0 = \frac{1}{2}$ ,

$$\begin{array}{cccccc} \epsilon & R(t) & H_0 t_0 & q_0 & \beta(t) & 1 + z \\ -1 & t^{1/3} & \frac{1}{3} & 2 & t & (1 + z)^4, \end{array} \quad (5.7)$$

$$\begin{array}{cccccc} +1 & t & 1 & 0 & t^{-1} & \text{const.}, \end{array} \quad (5.8)$$

$$\begin{array}{cccccc} -\frac{1}{2} & t^{1/2} & \frac{1}{2} & 1 & t^{1/2} & (1 + z)^2, \end{array} \quad (5.9)$$

$$\begin{array}{cccccc} +\frac{1}{2} & t^{5/6} & \frac{5}{6} & \frac{1}{3} & t^{-1/2} & (1 + z)^{2/5}, \end{array} \quad (5.10)$$

where  $t$  is normalized to  $t_0$  and  $R(t)$  to  $R_0$ . In spite of having  $q_0 = 0, \frac{1}{3}, 1$ , and 2, the four cases correspond to  $\bar{q}_0 = \frac{1}{2}$  and, as we shall see,  $\bar{q}_0$  and not  $q_0$  is the deceleration parameter determined from the  $m$  versus  $z$  relation (see [7.7] and [7.11]).

### c) The Full Solution

Equations (2.8)–(2.10) were solved numerically, and the results are presented in Tables 1–4. The procedure is as follows. Since the physically significant parameter is the curvature  $k$ , we first specify values of  $\bar{q}_0$  which, in spite of having lost its direct physical meaning in our interpretation, is nonetheless a convenient parameter whose values indicates whether  $k = -1, 0, +1$ , eq. (2.3).

Given a value of  $\bar{q}_0$  (first column), we then solve equations (2.14)–(2.16) with  $\beta = 1$ , so as to get  $\bar{H}_0 \bar{t}_0$  (third column). With this information we then compute  $H_0 t_0$  (eq. [5.1]), (fourth column) and finally  $q_0$ , equation (5.2) (second column). Equations (2.8)–(2.10) can then be solved. We present in columns (7) and (8) the function  $R(t)/R_0$  versus  $t/t_0$ , as well as the redshift  $z$ , equation (3.1) (sixth column), and  $z$ , equation (3.2) (fifth column). Finally, in column (9) we present  $\beta(t)$ , which can then be read against  $R/R_0$  or  $t/t_0$  or  $z$ .

The values corresponding to  $\bar{q}_0 = \frac{1}{2}$  ( $k = 0$ ), can be checked against the analytic expressions given by equations (5.3)–(5.10).

### d) Look-back Time

An important parameter in any cosmology is the look-back time, defined as

$$\tau = t_0 - t = (1 - t/t_0)t_0. \quad (5.11)$$

From column (8) of Tables 1–4, we can construct the function  $\tau/t_0$  versus  $z$ . For the four cases  $\epsilon = \pm 1, \pm \frac{1}{2}$ , the curves  $\tau/t_0$  versus  $z$  are presented in Figures 1, 2, 3, and 4, for two values of  $\bar{q}_0 = 0$  and 1, i.e., for an open and a closed universe. These figures should be compared with the one presented by Sandage for the case of ordinary cosmology (Sandage 1961b).

### e) Small Look-back Times

The case of small look-back times can be worked out explicitly, and it is therefore of interest because it yields a simple relation between the pairs of variables  $z$ ,  $H_0$  and  $z$ ,  $\bar{H}_0$ . From equation (4.1) we have

$$\beta(t) = 1 - (t_0 - t)h_0 - \frac{1}{2}Q_0 h_0^2 (t_0 - t)^2 + \dots \quad (5.12)$$

From (3.1) we obtain the expression for  $t_0 - t$ , namely

$$t_0 - t = \frac{z}{\bar{H}_0} [1 - (1 + \frac{1}{2}\bar{q}_0)z] + \dots,$$

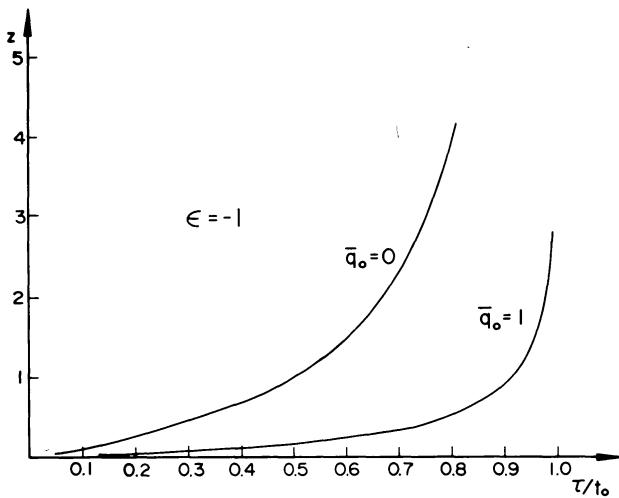


FIG. 1

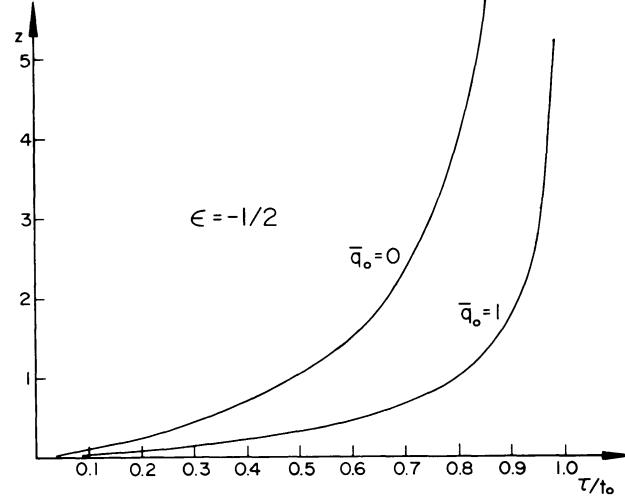


FIG. 2

FIG. 1.—Look-back time (normalized to today) versus  $z$  for  $\bar{q}_0 = 0$  and 1:  $\epsilon = -1$ FIG. 2.—Same as Fig. 1 for  $\epsilon = -\frac{1}{2}$

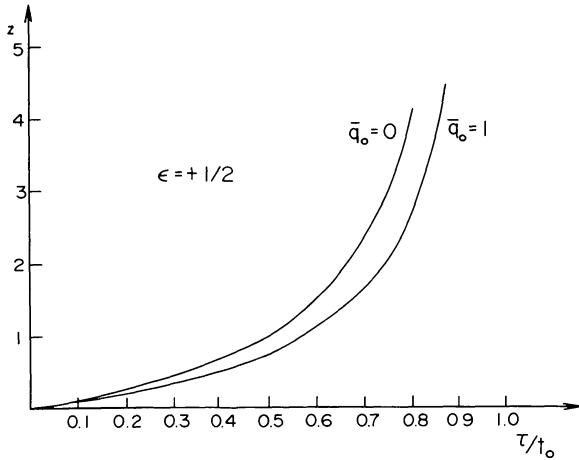


FIG. 3

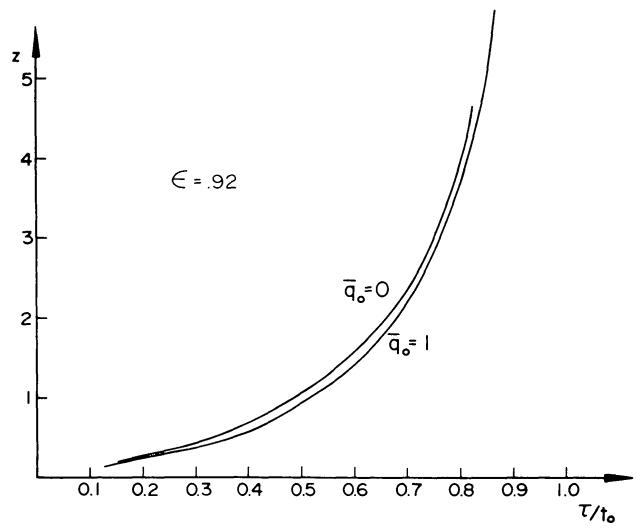


FIG. 4

FIG. 3.—Same as Fig. 1 for  $\epsilon = \frac{1}{2}$

FIG. 4.—Same as Fig. 1 for  $\epsilon = 0.92$

so that

$$\beta(t) = 1 - \frac{h_0}{H_0} z + \frac{h_0}{H_0} \left( 1 + \frac{1}{2}q_0 - \frac{1}{2}Q_0 \frac{h_0}{H_0} \right) z^2 + \dots \quad (5.13)$$

Substituting (5.13) into (3.3), we now obtain

$$z = z \left\{ 1 + \frac{h_0}{H_0} + \left[ \left( 1 + \frac{1}{2}Q_0 \right) \frac{h_0}{H_0} - \frac{1}{2}q_0 \right] \frac{h_0}{H_0} z + \dots \right\}, \quad (5.14)$$

which on the basis of both expressions (5.1) can be rewritten as

$$z = z \frac{\bar{H}_0}{H_0} + \frac{\epsilon}{2H_0 t_0} \left( q_0 + \frac{\epsilon - 1}{t_0 H_0} \right) z^2 + \dots \quad (5.15)$$

The range of validity of this approximation can be checked using the exact values of  $z$  versus  $z$  presented in the tables (and eq. [5.6] for the  $k = 0$  case). Some general relationship can already be reached at the level of (5.15). Since  $\bar{H}_0 = H_0 - \epsilon/t_0$  (eq. [2.2] with [4.1]),  $\bar{H}_0 > H_0$  for  $\epsilon < 0$  and so

$$z > z. \quad (5.16)$$

Conversely, for the cases  $\epsilon > 0$ ,  $\bar{H}_0 < H_0$  and so

$$z < z. \quad (5.17)$$

These two conclusions are borne out by the numerical values of  $z$  and  $z$  listed in Tables 1–4.

## VI. THE HORIZON

A concept of great importance in cosmology is that of the horizon—particle horizon and event horizon (Rindler 1956).

Following Rindler, we shall define the proper distance of the particle horizon (i.e., the farthest object from which we can receive signals) as

$$d_H(t) = R(t) \int_0^t \frac{dt}{R(t)}. \quad (6.1)$$

Since  $\beta dt = d\bar{t}$  and  $\beta R(t) = \bar{R}(\bar{t})$ , the integral can be transformed into an expression in Einstein units. The integration can then be performed. The final result, valid for any  $k$ , is

$$d_H(t) = R(t) \left( \frac{c}{\bar{H}_0 \bar{R}_0} \right) \frac{2}{(1 - 2\bar{q}_0)^{1/2}} \sinh^{-1} \left( \frac{1 - 2\bar{q}_0}{2\bar{q}_0} \frac{\bar{R}}{\bar{R}_0} \right)^{1/2}. \quad (6.2)$$

For  $2\bar{q}_0 < 1$ , it can be further transformed to  $k = -1$ :

$$\frac{d_H(t)}{R(t)} = \frac{c}{\bar{H}_0 \bar{R}_0} (1 - 2\bar{q}_0)^{-1/2} \psi. \quad (6.3)$$

Analogously, for  $2\bar{q}_0 > 1$ ,  $k = +1$ :

$$\frac{d_H(t)}{R(t)} = \frac{c}{\bar{H}_0 \bar{R}_0} (2\bar{q}_0 - 1)^{-1/2} \theta, \quad (6.4)$$

where we have used equations (2.9) and (2.10).

Finally, for  $k = 0$ ,

$$\frac{d_H(t)}{R(t)} = 2 \left( \frac{c}{\bar{H}_0 \bar{R}_0} \right) \left( \frac{\bar{R}}{\bar{R}_0} \right)^{1/2} = 2 \left( \frac{c}{\bar{H}_0 \bar{R}_0} \right) \beta^{1/2}(t) \left[ \frac{R(t)}{R_0} \right]^{1/2}. \quad (6.5)$$

#### a) Analytic Case: $k = 0$

Using equations (5.3) and (4.1) for  $R(t)$  and  $\beta(t)$  and (5.4), we may rewrite equation (6.5) as

$$d_H(t) = \left( \frac{2 + \epsilon}{1 - \epsilon} \right) \frac{c}{H_0} \left( \frac{t}{t_0} \right). \quad (6.6)$$

Explicitly,  $\epsilon = -1$ :

$$d_H(t) = \frac{3}{2}ct; \quad d_H(\bar{t}) = 3c\bar{t} = \frac{3ct\beta}{2} = \beta(t)d_H(t). \quad (6.7)$$

Analogously for  $\epsilon = \pm \frac{1}{2}$ :

$$d_H(\bar{t}) = \beta d_H(t), \quad (6.8)$$

where  $d_H(\bar{t}) = 3c\bar{t}$  corresponds to standard cosmology. The relation (6.7) or (6.8) is clearly valid also for the  $1 > \epsilon > 0$  case. We can therefore conclude that

$$\begin{aligned} \frac{d_H(\bar{t})}{d_H(t)} &< 1 \quad (\epsilon = -1, \epsilon = -\frac{1}{2}), \\ \frac{d_H(\bar{t})}{d_H(t)} &> 1 \quad (1 > \epsilon > 0). \end{aligned} \quad (6.9)$$

#### b) The $k = \pm 1$ Cases

Due to the implicit nature of the  $R(t)$  versus  $t$  function, we can only present asymptotic relations. Numerical values of  $d\bar{g}(t)$  can, however, be computed using the value of  $\bar{q}_0$  and  $R(t)/R_0$  given in the tables. For the  $k = -1$  case and for large development angles  $\psi$ , we obtain from (2.9)

$$\beta(t) \frac{R(t)}{R_0} \rightarrow \frac{q_0}{1 - 2\bar{q}_0} \frac{1}{2} e^\psi, \quad (6.10)$$

$$\int_0^t \beta(t) dt \rightarrow \frac{1}{\bar{H}_0} \frac{\bar{q}_0}{(1 - 2\bar{q}_0)^{3/2}} \frac{1}{2} e^\psi,$$

$$\frac{R(t)}{R_0} \rightarrow (1 - 2\bar{q}_0)^{1/2} \bar{H}_0 \left[ \int_0^t \beta(t) dt \right] / \beta(t). \quad (6.11)$$

We now have from (6.3)

$$d_H(t) = \frac{c}{\bar{H}_0} \left[ \frac{R(t)}{R_0} \right] \frac{1}{(1 - 2\bar{q}_0)^{1/2}} \psi \rightarrow \left[ c \int_0^t \beta(t) dt / \beta(t) \right] \psi; \quad (6.12)$$

or using (4.1),

$$d_H(t) \rightarrow \frac{ct}{1 - \epsilon} \psi \quad (6.13)$$

or

$$d_H(t) = \frac{1}{2}ct\psi = \beta^{-1}d_H(t) \quad (\epsilon = -1)$$

$$d_H(t) = \frac{3}{2}ct\psi = \beta^{-1}d_H(\bar{t}) \quad (\epsilon = -\frac{1}{2}). \quad (6.14)$$

An analogous relation holds true for the case  $0 > \epsilon > 1$ . The general conclusions (6.9) are therefore still valid.

## VII. THE $m$ - $z$ RELATION

We now proceed to derive the magnitude versus redshift relation making use of results obtained in Paper II. Radiation propagating freely through space can be represented by the energy-momentum tensor

$$T^{\mu\nu} = E k^\mu k^\nu,$$

where  $k^\mu$  is the tangent vector of the photon path. It has been shown (see eqs. [3.23] and [3.25] of Paper II) that along the beam of radiation,

$$EA\beta^{-\Pi_g} = \text{constant},$$

where  $A$  is the cross section of the beam. For radiation emerging radially from the source,  $A$  takes on the value of the area of concentric spherical surfaces. For a Robertson-Walker (R-W) metric

$$A = 4\pi r^2 R^2,$$

and hence

$$4\pi Er^2 R^2 \beta^{-\Pi_g} = \text{constant}. \quad (7.1)$$

Next, we note that the apparent luminosity, being equal to the observed flux, is given by

$$l = c\rho_\gamma = cT^{\mu\nu} u_\mu u_\nu = cE(u_\mu k^\mu)^2 = cE(v^2/\beta^2),$$

where, in the last equality, equation (6.5) of Paper II has been used. Thus we can write (7.1) as

$$\frac{4\pi}{c} l v^{-2} \beta^{2-\Pi_g} r^2 R^2 = \text{constant}. \quad (7.2)$$

But, as we approach the source,

$$\lim_{r \rightarrow 0} 4\pi r^2 R^2 l = L(t), \quad (7.3)$$

where  $L(t)$  is the intrinsic luminosity at the time of emission,  $t$ . The above two equations yield

$$l_{\text{obs}} = \frac{L(t)}{4\pi} \frac{1}{r_{\text{obs}}^2 R_{\text{obs}}^2} \left( \frac{\beta_{\text{em}}}{\beta_{\text{obs}}} \right)^{2-\Pi_g} \left( \frac{v_{\text{obs}}}{v_{\text{em}}} \right)^2 = \frac{L(t)}{4\pi r_{\text{obs}}^2 R_{\text{obs}}^2} \frac{1}{(1+z)^2} \left( \frac{\beta_{\text{em}}}{\beta_{\text{obs}}} \right)^{2-\Pi_g} \quad (7.4a)$$

where  $r_{\text{obs}}$  is the radial coordinate distance from the emitter to the observer. If  $\beta$  is normalized such that  $\beta = 1$  at the point of observation, we finally have

$$l = \frac{L(t)}{4\pi r_e^2 R_0^2} \frac{1}{(1+z)^2} \beta^2 G(\beta), \quad (7.4b)$$

where we have dropped the subscript "obs" from  $l$  and called  $r_{\text{obs}} = r_e$ , and  $R_{\text{obs}} = R_0 \equiv R(t_0)$ . It is easy to reduce (7.4b) to the case of ordinary cosmology: in fact, it is sufficient to put  $\beta = 1$  and  $G(\beta) = 1$ . It is in fact understood that  $G(\beta)$  is normalized to today's value. We want to note at this point that the gauge condition  $G\beta^2 = 1$ , which makes  $\rho_\gamma$  (eq. [2.7]) independent of  $\beta$ , also makes  $l$  independent of the scale factor  $\beta$ , at least explicitly. An alternative derivation of (7.4b) will be presented in Appendix B.

The comoving radial coordinate  $r_e$  of the light source is obtained by solving the equation

$$\int_0^{r_e} \frac{dr}{(1-kr^2)^{1/2}} = \int_{t_0}^{t_e} \frac{cdt}{R(t)}.$$

Since the integrand of the right-hand side can be transformed to Einstein units by the change of variables  $\beta dt = d\bar{t}$ ,  $\beta R(t) = \bar{R}(\bar{t})$ , and since  $\bar{R}(\bar{t})$  satisfies the ordinary Einstein equation, we obtain, upon integrating, the following result:

$$\frac{\bar{H}_0 \bar{R}_0}{c} (1+z)r_e = F(z, \bar{q}_0) \quad (7.5)$$

with

$$F(z, \bar{q}_0) = \bar{q}_0^{-2} \{ \bar{q}_0 z + (\bar{q}_0 - 1)[(1+2\bar{q}_0 z)^{1/2} - 1] \}, \quad (7.6)$$

or alternatively

$$F(z, \bar{q}_0) = z \left[ 1 + z \frac{1 - \bar{q}_0}{1 + \bar{q}_0 z + (1 + 2\bar{q}_0 z)^{1/2}} \right], \quad (7.6a)$$

where (7.6a) is an algebraic manipulation of (7.6) (Terrell 1977) which facilitates a great deal the computation in the limit of  $\bar{q}_0 \rightarrow 0$ .

TABLE 5,  $\epsilon = -1$ 

$e = -2$			$e = -1$			$e = 0$			$e = 1$			$e = +2$		
$\bar{q}_0$	$z$	$m$	$\theta_m$	$m$	$\theta_m$									
0.00														
6.559	7.970	4.549	10.166	4.549	12.362	4.549	14.558	4.549	16.755	4.549	18.949	4.549	21.143	4.549
4.164	5.899	6.667	7.681	6.667	9.464	6.667	11.246	6.667	13.029	6.667	15.029	6.667	17.029	6.667
3.170	4.735	8.271	6.285	8.271	7.835	8.271	9.386	8.271	10.936	8.271	12.936	8.271	14.936	8.271
2.592	3.918	9.628	5.307	9.628	6.695	9.628	8.084	9.628	9.472	9.628	10.472	9.628	11.472	9.628
2.203	3.288	10.837	4.551	10.837	5.815	10.837	7.079	10.837	10.837	8.343	10.837	8.343	10.837	8.343
1.917	2.772	11.949	3.934	11.949	5.097	11.949	6.259	11.949	7.422	11.949	8.422	11.949	9.422	11.949
1.697	2.334	12.993	3.411	12.993	4.488	12.993	5.565	12.993	6.642	12.993	7.642	12.993	8.642	12.993
1.520	1.952	13.990	2.956	13.990	3.959	13.990	4.962	13.990	5.966	13.990	6.966	13.990	7.966	13.990
1.374	1.613	14.955	2.551	14.955	3.490	14.955	4.428	14.955	5.367	14.955	6.367	14.955	7.367	14.955
1.250	1.306	15.898	2.187	15.898	3.067	15.898	3.948	15.898	4.828	15.898	5.828	15.898	6.828	15.898
1.144	1.025	16.829	1.854	16.829	2.682	16.829	3.510	16.829	4.338	16.829	5.338	16.829	6.338	16.829
1.052	0.766	17.755	1.546	17.755	2.327	17.755	3.107	17.755	3.887	17.755	4.887	17.755	5.887	17.755
0.971	0.523	18.684	1.260	18.684	1.996	18.684	2.733	18.684	3.469	18.684	4.469	18.684	5.469	18.684
0.898	0.295	19.620	0.991	19.620	1.687	19.620	2.383	19.620	3.079	19.620	4.079	19.620	5.079	19.620
0.055	-4.025	68.050	-3.868	68.050	-3.712	68.050	-3.556	68.050	-3.400	68.050	-3.400	68.050	-3.400	68.050
0.118	-4.623	85.412	-4.502	85.412	-4.381	85.412	-4.260	85.412	-4.139	85.412	-4.139	85.412	-4.139	85.412
0.085	-5.353	114.214	-5.265	114.214	-5.176	114.214	-5.088	114.214	-5.000	114.214	-5.000	114.214	-5.000	114.214
0.054	-6.331	171.649	-6.273	171.649	-6.216	171.649	-6.159	171.649	-6.102	171.649	-6.102	171.649	-6.102	171.649
0.026	-7.926	343.662	-7.898	343.662	-7.870	343.662	-7.843	343.662	-7.815	343.662	-7.815	343.662	-7.815	343.662
0.100														
3.979	1.917	13.886	5.930	13.886	9.943	13.886	13.955	13.886	17.968	13.886	18.968	13.886	19.968	13.886
3.060	1.821	13.011	5.185	13.011	8.549	13.011	11.913	13.011	15.277	13.011	16.277	13.011	17.277	13.011
2.497	1.659	12.881	4.558	12.881	7.457	12.881	13.355	12.881	13.254	12.881	13.254	12.881	13.254	12.881
2.101	1.458	13.141	3.993	13.141	6.527	13.141	9.062	13.141	11.597	13.141	13.597	13.141	14.597	13.141
1.799	1.229	13.661	3.464	13.661	5.699	13.661	7.934	13.661	13.661	13.661	13.661	13.661	13.661	13.661
1.557	0.979	14.388	2.958	14.388	4.937	14.388	6.917	14.388	8.896	14.388	10.896	14.388	12.896	14.388
1.354	0.710	15.302	2.466	15.302	4.222	15.302	5.978	15.302	7.733	15.302	9.733	15.302	11.733	15.302
1.180	0.425	16.406	1.982	16.406	3.538	16.406	5.094	16.406	6.651	16.406	8.651	16.406	10.651	16.406
1.028	0.124	17.716	1.500	17.716	2.876	17.716	4.252	17.716	5.628	17.716	7.628	17.716	9.628	17.716
0.892	-0.196	19.269	1.015	19.269	2.226	19.269	3.437	19.269	4.648	19.269	6.648	19.269	8.648	19.269
0.769	-0.534	21.121	0.524	21.121	1.582	21.121	2.640	21.121	3.698	21.121	5.698	21.121	7.698	21.121
0.656	-0.897	23.369	0.019	23.369	0.934	23.369	1.850	23.369	2.766	23.369	4.766	23.369	6.766	23.369
0.553	-1.288	26.165	-0.507	26.165	0.275	26.165	1.057	26.165	1.838	26.165	2.838	26.165	3.838	26.165
0.457	-1.719	29.774	-1.064	29.774	-0.409	29.774	0.246	29.774	0.901	29.774	1.901	29.774	2.901	29.774
0.368	-2.204	34.677	-1.669	34.677	-1.134	34.677	-0.600	34.677	-0.065	34.677	-0.065	34.677	-0.065	34.677
0.285	-2.769	41.844	-2.350	41.844	-1.930	41.844	-1.511	41.844	-1.091	41.844	-1.091	41.844	-1.091	41.844
0.207	-3.466	53.551	-3.157	53.551	-2.848	53.551	-2.539	53.551	-2.230	53.551	-2.230	53.551	-2.230	53.551
0.133	-4.410	76.639	-4.207	76.639	-4.005	76.639	-3.802	76.639	-3.599	76.639	-3.599	76.639	-3.599	76.639
0.065	-5.968	145.336	-5.869	145.336	-5.769	145.336	-5.669	145.336	-5.569	145.336	-5.569	145.336	-5.569	145.336
0.500														
3.472	-3.243	80.916	1.636	80.916	6.515	80.916	11.394	80.916	16.273	80.916	21.273	80.916	26.273	80.916
1.789	-2.918	55.011	0.423	55.011	3.764	55.011	7.105	55.011	10.446	55.011	15.011	55.011	20.011	55.011
1.497	-2.878	51.120	0.102	51.120	3.083	51.120	6.064	51.120	9.045	51.120	12.045	51.120	15.045	51.120
1.338	-2.866	49.183	-0.100	49.183	2.666	49.183	5.432	49.183	8.198	49.183	11.198	49.183	14.198	49.183
1.230	-2.863	47.982	-0.251	47.982	2.362	47.982	4.975	47.982	7.587	47.982	10.587	47.982	13.587	47.982
1.150	-2.866	47.157	-0.372	47.157	2.121	47.157	4.614	47.157	7.107	47.157	9.707	47.157	11.707	47.157
1.086	-2.870	46.558	-0.475	46.558	1.920	46.558	4.316	46.558	6.711	46.558	9.711	46.558	12.711	46.558
0.361	-3.526	50.852	-2.523	50.852	-1.519	50.852	-0.516	50.852	-0.488	50.852	-0.488	50.852	-0.488	50.852
0.334	-3.611	52.348	-2.672	52.348	-1.734	52.348	-0.795	52.348	-0.143	52.348	-0.143	52.348	-0.143	52.348
0.310	-3.695	53.932	-2.814	53.932	-1.934	53.932	-1.053	53.932	-0.172	53.932	-0.172	53.932	-0.172	53.932
0.290	-3.778	55.605	-2.950	55.605	-2.122	55.605	-1.294	55.605	-0.466	55.605	-0.466	55.605	-0.466	55.605
0.271	-3.862	57.371	-3.082	57.371	-2.301	57.371	-1.521	57.371	-0.741	57.371	-0.741	57.371	-0.741	57.371
0.254	-3.946	59.236	-3.210	59.236	-2.473	59.236	-1.737	59.236	-1.000	59.236	-1.000	59.236	-1.000	59.236
0.238	-4.031	61.204	-3.335	61.204	-2.639	61.204	-1.943	61.204	-1.247	61.204	-1.247	61.204	-1.247	61.204
0.059	-6.402	168.679	-6.216	168.679	-6.330	168.679	-5.844	168.679	-5.659	168.679	-5.659	168.679	-5.659	168.679
0.047	-6.847	205.826	-6.698	205.826	-6.549	205.826	-6.400	205.826	-6.251	205.826	-6.251	205.826	-6.251	205.826
0.036	-7.386	262.466	-7.272	262.466	-7.157	262.466	-7.043	262.466	-6.929	262.466	-6.929	262.466	-6.929	262.466
0.025	-8.080	359.489	-7.998	359.489	-7.916	359.489	-7.834	359.489	-7.753	359.489	-7.753	359.489	-7.753	359.489
0.016	-9.069	564.208	-9.018	564.208	-8.967	564.208	-8.915	564.208	-8.864	564.208	-8.864	564.208	-8.864	564.208
1.000														
2.822	-6.409	265.657	-1.628	265.657	3.153	265.657	7.934	265.657	12.716	265.657	12.716	265.657	12.716	265.657
1.719	-6.057	190.769	-2.388	190.769	1.281	190.769	4.950	190.769	8.619	190.769	8.619	190.769	8.619	190.769
1.227	-5.870	158.780	-2.855	158.780	0.160	158.780	3.175	158.780	6.190	158.780	6.190	158.780	6.190	158.780
0.936	-5.758	141.075	-3.208	141.075	-0.658	141.075	1.892	141.075	4.442	141.07				

TABLE 6,  $\epsilon = -\frac{1}{2}$ 

$\bar{q}_0$	$z$	$m$	$\theta m$	$m$	$\theta m$	$e = -2$	$e = -1$	$e = 0$	$e = 1$	$e = +2$							
0.00	1.835	9.327	6.696	1.0.791	6.696	1.2.255	6.696	1.3.719	6.696	1.5.183	6.696	1.0.791	6.696	1.1.871	6.696	1.2.805	
7.926	7.118	8.642	8.306	8.642	9.494	8.642	1.0.683	8.642	1.1.683	8.642	1.2.683	8.642	1.3.683	8.642	1.4.683	8.642	
5.712	5.876	9.985	6.910	9.985	7.943	9.985	8.977	9.985	1.0.010	9.985	1.1.010	9.985	1.2.010	9.985	1.3.010	9.985	
4.501	5.006	11.059	5.932	11.059	6.857	11.059	7.783	11.059	8.708	11.059	9.708	11.059	10.708	11.059	11.708	11.059	
3.721	4.334	11.981	5.176	11.981	6.019	11.981	6.861	11.981	7.704	11.981	8.704	11.981	9.704	11.981	10.704	11.981	
3.168	3.784	12.805	4.559	12.805	5.334	12.805	6.109	12.805	6.884	12.805	7.884	12.805	8.884	12.805	9.884	12.805	
2.754	3.318	13.564	4.036	13.564	4.754	13.564	5.472	13.564	6.190	13.564	6.910	13.564	7.190	13.564	8.190	13.564	
2.429	2.911	14.278	3.580	14.278	4.249	14.278	4.918	14.278	5.587	14.278	6.587	14.278	7.587	14.278	8.587	14.278	
2.166	2.550	14.961	3.176	14.961	3.802	14.961	4.427	14.961	5.053	14.961	5.624	14.961	5.572	14.961	5.961	14.961	
1.949	2.224	15.624	2.811	15.624	3.398	15.624	3.985	15.624	4.572	15.624	5.135	15.624	5.572	15.624	6.135	15.624	
1.765	1.926	16.276	2.478	16.276	3.030	16.276	3.582	16.276	4.135	16.276	4.735	16.276	5.135	16.276	5.735	16.276	
1.607	1.651	16.921	2.171	16.921	2.691	16.921	3.211	16.921	3.732	16.921	4.311	16.921	4.732	16.921	5.311	16.921	
1.471	1.394	17.568	1.885	17.568	2.376	17.568	2.867	17.568	3.358	17.568	3.867	17.568	4.358	17.568	4.867	17.568	
1.350	1.152	18.220	1.616	18.220	2.080	18.220	2.544	18.220	3.008	18.220	3.544	18.220	3.908	18.220	4.308	18.220	
0.211	-3.348	53.544	-3.244	53.544	-3.140	53.544	-3.035	53.544	-2.931	53.544	-2.831	53.544	-2.731	53.544	-2.631	53.544	
0.160	-3.958	66.486	-3.877	66.486	-3.797	66.486	-3.716	66.486	-3.635	66.486	-3.563	66.486	-3.493	66.486	-3.423	66.486	
0.114	-4.699	88.013	-4.640	88.013	-4.581	88.013	-4.522	88.013	-4.463	88.013	-4.393	88.013	-4.323	88.013	-4.253	88.013	
0.073	-5.687	131.018	-5.649	131.018	-5.611	131.018	-5.572	131.018	-5.534	131.018	-5.494	131.018	-5.454	131.018	-5.414	131.018	
0.035	-7.292	259.959	-7.274	259.959	-7.255	259.959	-7.236	259.959	-7.218	259.959	-7.188	259.959	-7.168	259.959	-7.148	259.959	
0.100	1.069	4.107	32.141	6.733	32.141	9.458	32.141	12.133	32.141	14.808	32.141	14.808	32.141	32.141	32.141	32.141	
1.044	3.795	24.674	6.038	24.674	8.280	24.674	10.523	24.674	12.766	24.674	12.766	24.674	12.766	24.674	24.674	24.674	
7.514	3.478	21.178	5.411	21.178	7.343	21.178	9.276	21.178	11.208	21.178	11.208	21.178	11.208	21.178	21.178	21.178	
5.752	3.155	19.322	4.845	19.322	6.535	19.322	8.225	19.322	9.915	19.322	9.915	19.322	9.915	19.322	10.915	19.322	
4.560	2.827	18.320	4.317	18.320	5.807	18.320	7.297	18.320	8.787	18.320	8.787	18.320	8.787	18.320	9.787	18.320	
3.695	2.491	17.838	3.811	17.838	5.130	17.838	6.450	17.838	7.770	17.838	7.770	17.838	7.770	17.838	8.770	17.838	
3.036	2.148	17.713	3.319	17.713	4.489	17.713	5.660	17.713	6.830	17.713	6.830	17.713	6.830	17.713	7.830	17.713	
2.516	1.797	17.863	2.834	17.863	3.872	17.863	4.910	17.863	5.947	17.863	5.947	17.863	5.947	17.863	6.947	17.863	
2.094	1.435	18.251	2.352	18.251	3.270	18.251	4.187	18.251	5.105	18.251	5.105	18.251	5.105	18.251	6.105	18.251	
1.743	1.061	18.869	1.868	18.869	2.675	18.869	3.483	18.869	4.290	18.869	4.290	18.869	4.290	18.869	5.290	18.869	
1.447	0.671	19.735	1.376	19.735	2.082	19.735	2.787	19.735	3.493	19.735	3.493	19.735	3.493	19.735	4.493	19.735	
1.194	0.261	20.901	0.872	20.901	1.482	20.901	2.092	20.901	2.703	20.901	2.703	20.901	2.703	20.901	3.703	20.901	
0.974	-0.175	22.459	0.346	22.459	0.867	22.459	1.388	22.459	1.909	22.459	1.909	22.459	1.909	22.459	2.909	22.459	
0.781	-0.648	24.582	-0.211	24.582	0.226	24.582	0.662	24.582	1.099	24.582	1.099	24.582	1.099	24.582	1.499	24.582	
0.612	-1.173	27.590	-0.816	27.590	-0.460	27.590	-0.103	27.590	0.253	27.590	0.253	27.590	0.253	27.590	0.590	27.590	
0.461	-1.776	32.137	-1.497	32.137	-1.217	32.137	-0.937	32.137	-0.658	32.137	-0.658	32.137	-0.658	32.137	-1.337	32.137	
0.327	-2.510	39.757	-2.304	39.757	-2.098	39.757	-1.892	39.757	-1.686	39.757	-1.686	39.757	-1.686	39.757	-1.486	39.757	
0.206	-3.489	55.069	-3.354	55.069	-3.219	55.069	-3.084	55.069	-2.949	55.069	-2.949	55.069	-2.949	55.069	-2.669	55.069	
0.098	-5.082	101.185	-5.016	101.185	-4.949	101.185	-4.883	101.185	-4.816	101.185	-4.816	101.185	-4.816	101.185	-4.616	101.185	
0.500	7.298	-0.279	81.090	2.019	81.090	4.316	81.090	6.614	81.090	8.911	81.090	8.911	81.090	8.911	81.090	8.911	81.090
5.437	-0.367	65.494	1.655	65.494	3.677	65.494	5.698	65.494	7.720	65.494	7.720	65.494	7.720	65.494	9.720	65.494	
4.582	-0.429	58.442	1.438	58.442	3.305	58.442	5.172	58.442	7.039	58.442	7.039	58.442	7.039	58.442	8.039	58.442	
4.053	-0.479	54.144	1.280	54.144	3.039	54.144	4.798	54.144	6.557	54.144	6.557	54.144	6.557	54.144	7.557	54.144	
3.681	-0.522	51.159	1.154	51.159	2.830	51.159	4.506	51.159	6.181	51.159	6.181	51.159	6.181	51.159	7.159	51.159	
3.399	-0.560	48.927	1.048	48.927	2.657	48.927	4.265	48.927	5.873	48.927	5.873	48.927	5.873	48.927	6.873	48.927	
3.174	-0.595	47.176	0.957	47.176	2.508	47.176	4.060	47.176	5.611	47.176	5.611	47.176	5.611	47.176	6.611	47.176	
0.852	-1.686	34.599	-1.017	34.599	-0.349	34.599	-0.320	34.599	-0.298	34.599	-0.298	34.599	-0.298	34.599	-0.269	34.599	
0.779	-1.793	34.914	-1.167	34.914	-0.541	34.914	0.084	34.914	0.710	34.914	0.710	34.914	0.710	34.914	1.710	34.914	
0.717	-1.896	35.336	-1.309	35.336	-0.722	35.336	-0.135	35.336	-0.452	35.336	-0.452	35.336	-0.452	35.336	-0.336	35.336	
0.663	-1.997	35.852	-1.445	35.852	-0.893	35.852	-0.341	35.852	-0.211	35.852	-0.211	35.852	-0.211	35.852	-0.152	35.852	
0.615	-2.097	36.452	-1.577	36.452	-1.056	36.452	-0.536	36.452	-0.016	36.452	-0.016	36.452	-0.016	36.452	-0.232	36.452	
0.572	-2.196	37.133	-1.705	37.133	-1.214	37.133	-0.723	37.133	-0.339	37.133	-0.339	37.133	-0.339	37.133	-0.133	37.133	
0.533	-2.294	37.891	-1.830	37.891	-1.366	37.891	-0.902	37.891	-0.438	37.891	-0.438	37.891	-0.438	37.891	-0.211	37.891	
0.121	-4.835	89.294	-4.711	89.294	-4.587	89.294	-4.463	89.294	-4.339	89.294	-4.339	89.294	-4.339	89.294	-4.294	89.294	
0.096	-5.292	107.729	-5.193	107.729	-5.093	107.729	-4.994	107.729	-4.895	107.729	-4.895	107.729	-4.895	107.729	-4.729	107.729	
0.073	-5.843	135.924	-5.767	135.924	-5.690	135.924	-5.614	135.924	-5.538	135.924	-5.538	135.924	-5.538	135.924	-5.420	135.924	
0.052	-6.548	184.320	-6.493	184.320	-6.438	184.320	-6.384	184.320	-6.329	184.320	-6.329	184.320	-6.329	184.320	-6.230	184.320	
0.032	-7.547	286.573	-7.513	286.573	-7.479	286.573	-7.444	286.573	-7.399	286.573	-7.399	286.573	-7.399	286.573	-7.410	286.573	
1.000	8.611	-2.200	200.462	0.394	200.462	2.989	200.462	5.583	200.462	8.177</td							

TABLE 7,  $\epsilon = \frac{1}{2}$ 

$\bar{q}_e$	$e = -2$	$e = -1$	$e = 0$	$e = 1$	$e = +2$						
	$z$	$m$	$\theta_m$	$m$	$\theta_m$	$m$	$\theta_m$	$m$	$\theta_m$	$m$	$\theta_m$
0.0											
2.863	2.284	22.792	1.550	22.792	0.817	22.792	0.083	22.792	-0.651	22.792	
2.715	2.169	22.667	1.457	22.667	0.744	22.667	0.032	22.667	-0.681	22.667	
2.575	2.055	22.561	1.363	22.561	0.671	22.561	-0.020	22.561	-0.712	22.561	
2.445	1.941	22.476	1.279	22.476	0.598	22.476	-0.073	22.476	-0.745	22.476	
2.320	1.828	22.410	1.176	22.410	0.525	22.410	-0.127	22.410	-0.778	22.410	
2.203	1.715	22.363	1.083	22.363	0.451	22.363	-0.181	22.363	-0.813	22.363	
2.091	1.602	22.336	0.989	22.336	0.376	22.336	-0.236	22.336	-0.849	22.336	
1.985	1.489	22.329	0.895	22.329	0.301	22.329	-0.292	22.329	-0.886	22.329	
1.885	1.376	22.342	0.801	22.342	0.226	22.342	-0.349	22.342	-0.924	22.342	
1.789	1.263	22.376	0.706	22.376	0.149	22.376	-0.407	22.376	-0.964	22.376	
1.698	1.150	22.430	0.611	22.430	0.072	22.430	-0.467	22.430	-1.005	22.430	
1.612	1.036	22.507	0.515	22.507	-0.006	22.507	-0.527	22.507	-1.048	22.507	
1.529	0.922	22.606	0.419	22.606	-0.085	22.606	-0.589	22.606	-1.093	22.606	
1.451	0.808	22.729	0.321	22.729	-0.165	22.729	-0.652	22.729	-1.139	22.729	
0.306	-2.571	41.911	-2.716	41.911	-2.861	41.911	-3.006	41.911	-3.151	41.911	
0.235	-3.149	50.264	-3.263	50.264	-3.378	50.264	-3.492	50.264	-3.606	50.264	
0.169	-3.864	64.360	-3.949	64.360	-4.033	64.360	-4.118	64.360	-4.203	64.360	
0.108	-4.832	92.795	-4.888	92.795	-4.944	92.795	-5.000	92.795	-5.055	92.795	
0.052	-6.423	178.543	-6.450	178.543	-6.478	178.543	-6.505	178.543	-6.533	178.543	
0.100											
9.478	4.207	37.116	2.797	37.116	1.387	37.116	-0.022	37.116	-1.432	37.116	
8.131	3.931	34.573	2.606	34.573	1.280	34.573	-0.045	34.573	-1.371	34.573	
6.974	3.652	32.350	2.410	32.350	1.167	32.350	-0.076	32.350	-1.319	32.350	
5.976	3.369	30.414	2.207	30.414	1.046	30.414	-0.116	30.414	-1.278	30.414	
5.115	3.079	28.740	1.998	28.740	0.916	28.740	-0.165	28.740	-1.247	28.740	
4.369	2.783	27.308	1.780	27.308	0.777	27.308	-0.225	27.308	-1.228	27.308	
3.721	2.478	26.107	1.553	26.107	0.628	26.107	-0.297	26.107	-1.222	26.107	
3.159	2.163	25.130	1.314	25.130	0.466	25.130	-0.382	25.130	-1.231	25.130	
2.669	1.835	24.378	1.062	24.378	0.289	24.378	-0.484	24.378	-1.257	24.378	
2.241	1.492	23.863	0.793	23.863	0.095	23.863	-0.603	23.863	-1.302	23.863	
1.867	1.129	23.611	0.504	23.611	-0.120	23.611	-0.745	23.611	-1.37u	23.611	
1.539	0.742	23.667	0.190	23.667	-0.362	23.667	-0.915	23.667	-1.467	23.667	
1.251	0.323	24.113	-0.157	24.113	-0.638	24.113	-1.118	24.113	-1.599	24.113	
0.999	-0.138	25.39u	-0.548	25.39u	-0.957	25.39u	-1.367	25.39u	-1.777	25.39u	
0.776	-0.658	26.864	-0.998	26.864	-1.337	26.864	-1.677	26.864	-2.017	26.864	
0.581	-1.265	29.972	-1.535	29.972	-1.805	29.972	-2.076	29.972	-2.346	29.972	
0.408	-2.009	35.680	-2.211	35.680	-2.413	35.680	-2.614	35.680	-2.816	35.680	
0.255	-3.007	47.796	-3.141	47.796	-3.275	47.796	-3.409	47.796	-3.542	47.796	
0.120	-4.628	85.386	-4.694	85.386	-4.761	85.386	-4.828	85.386	-4.894	85.386	
0.500											
2.164	1.062	26.432	0.312	26.432	-0.438	26.432	-1.189	26.432	-1.939	26.432	
2.154	1.054	26.412	0.306	26.412	-0.443	26.412	-1.191	26.412	-1.939	26.412	
2.143	1.045	26.393	0.299	26.393	-0.447	26.393	-1.193	26.393	-1.939	26.393	
2.133	1.037	26.373	0.293	26.373	-0.451	26.373	-1.195	26.373	-1.939	26.354	
2.123	1.029	26.354	0.287	26.354	-0.455	26.354	-1.197	26.354	-1.939	26.335	
2.112	1.020	26.335	0.280	26.335	-0.459	26.335	-1.199	26.335	-1.939	26.335	
2.102	1.012	26.317	0.274	26.317	-0.463	26.317	-1.201	26.317	-1.938	26.317	
1.488	0.395	25.663	-0.199	25.663	-0.792	25.663	-1.386	25.663	-1.980	25.663	
1.418	0.307	25.676	-0.268	25.676	-0.843	25.676	-1.418	25.676	-1.993	25.676	
1.351	0.219	25.715	-0.338	25.715	-0.895	25.715	-1.452	25.715	-2.009	25.715	
1.287	0.129	25.780	-0.410	25.780	-0.948	25.780	-1.487	25.780	-2.026	25.780	
1.226	0.039	25.873	-0.482	25.873	-1.003	25.873	-1.525	25.873	-2.046	25.873	
1.167	-0.052	25.993	-0.556	25.993	-1.060	25.993	-1.564	25.993	-2.067	25.993	
1.111	-0.145	26.143	-0.631	26.143	-1.118	26.143	-1.605	26.143	-2.091	26.143	
0.298	-2.746	43.867	-2.916	43.867	-3.086	43.867	-3.256	43.867	-3.426	43.867	
0.237	-3.219	50.992	-3.358	50.992	-3.496	50.992	-3.635	50.992	-3.774	50.992	
0.181	-3.786	62.036	-3.895	62.036	-4.003	62.036	-4.112	62.036	-4.220	62.036	
0.129	-4.508	81.173	-4.587	81.173	-4.666	81.173	-4.745	81.173	-4.823	81.173	
0.080	-5.525	121.868	-5.575	121.868	-5.625	121.868	-5.675	121.868	-5.725	121.868	
1.000											
4.491	1.962	36.271	0.807	36.271	-0.347	36.271	-1.501	36.271	-2.655	36.271	
3.914	1.761	34.116	0.681	34.116	-0.399	34.116	-1.479	34.116	-2.559	34.116	
3.410	1.555	32.291	0.548	32.291	-0.460	32.291	-1.467	32.291	-2.475	32.291	
2.969	1.344	30.764	0.407	30.764	-0.530	30.764	-1.466	30.764	-2.403	30.764	
2.582	1.126	29.510	0.259	29.510	-0.609	29.510	-1.477	29.510	-2.345	29.510	
2.241	0.901	28.513	0.100	28.513	-0.700	28.513	-1.501	28.513	-2.301	28.513	
1.939	0.666	27.768	-0.069	27.768	-0.804	27.768	-1.538	27.768	-2.273	27.768	
1.673	0.420	27.274	-0.250	27.274	-0.921	27.274	-1.592	27.274	-2.262	27.274	
1.436	0.161	27.045	-0.447	27.045	-1.055	27.045	-1.663	27.045	-2.271	27.045	
1.225	-0.115	27.107	-0.662	27.107	-1.208	27.107	-1.755	27.107	-2.302	27.107	
1.036	-0.412	27.509	-0.898	27.509	-1.385	27.509	-1.871	27.509	-2.356	27.509	
0.868	-0.733	28.331	-1.161	28.331	-1.589	28.331	-2.017	28.331	-2.445	28.331	
0.716	-1.088	29.707	-1.459	29.707	-1.829	29.707	-2.200	29.707	-2.570	29.707	
0.580	-1.486	31.867	-1.801	31.867	-2.115	31.867	-2.429	31.867	-2.743	31.867	
0.458	-1.945	35.235	-2.204	35.235	-2.463	35.235	-2.722	35.235	-2.981	35.235	
0.348	-2.490	40.665	-2.695	40.665	-2.901	40.665	-3.106	40.665	-3.311	40.665	
0.248	-3.175	50.159	-3.327	50.159	-3.480	50.159	-3.632	50.159	-3.784	50.159	
0.157	-4.114	69.738	-4.215	69.738	-4.315	69.738	-4.416	69.738	-4.516	69.738	
0.075	-5.677	129.520	-5.727	129.520	-5.777	129.520	-5.826	129.520	-5.876	129.520	

TABLE 8,  $\epsilon = 0.92$ 

$\bar{q}_o$	$z$	$m$	$\theta_m$	$e = -2$	$e = -1$	$e = 0$	$e = 1$	$e = +2$								
u.uu																
4.676	3.090	28.007	1.356	28.007	-0.378	28.007	-2.113	28.007	-3.847	28.007	-2.847	27.571	27.571	27.571	28.007	
4.466	3.001	27.571	1.304	27.571	-0.392	27.571	-2.089	27.571	-3.785	27.571	-3.785	27.571	27.571	27.571	27.571	
4.263	2.911	27.157	1.252	27.157	-0.417	27.157	-2.066	27.157	-3.724	27.157	-3.724	27.157	27.157	27.157	27.157	
4.069	2.820	26.764	1.199	26.764	-0.422	26.764	-2.044	26.764	-3.665	26.764	-3.665	26.764	26.764	26.764	26.764	
3.882	2.728	26.392	1.144	26.392	-0.439	26.392	-2.023	26.392	-3.607	26.392	-3.607	26.392	26.392	26.392	26.392	
3.703	2.636	26.040	1.089	26.040	-0.457	26.040	-2.004	26.040	-3.550	26.040	-3.550	26.040	26.040	26.040	26.040	
3.531	2.542	25.709	1.033	25.709	-0.477	25.709	-1.986	25.709	-3.495	25.709	-3.495	25.709	25.709	25.709	25.709	
3.365	2.447	25.399	9.975	25.399	-0.497	25.399	-1.969	25.399	-3.441	25.399	-3.441	25.399	25.399	25.399	25.399	
3.206	2.351	25.109	9.916	25.109	-0.519	25.109	-1.954	25.109	-3.389	25.109	-3.389	25.109	25.109	25.109	25.109	
3.054	2.254	24.840	9.856	24.840	-0.542	24.840	-1.940	24.840	-3.338	24.840	-3.338	24.840	24.840	24.840	24.840	
2.907	2.156	24.591	9.795	24.591	-0.566	24.591	-1.928	24.591	-3.289	24.591	-3.289	24.591	24.591	24.591	24.591	
2.766	2.057	24.363	9.732	24.363	-0.593	24.363	-1.917	24.363	-3.242	24.363	-3.242	24.363	24.363	24.363	24.363	
2.631	1.956	24.157	9.668	24.157	-0.620	24.157	-1.908	24.157	-3.196	24.157	-3.196	24.157	24.157	24.157	24.157	
2.500	1.853	23.971	9.602	23.971	-0.649	23.971	-1.901	23.971	-3.152	23.971	-3.152	23.971	23.971	23.971	23.971	
3.521	-1.431	31.161	-1.851	31.161	-2.269	31.161	-2.688	31.161	-3.107	31.161	-3.107	31.161	31.161	31.161	31.161	
3.397	-2.015	35.900	-2.345	35.900	-2.683	35.900	-3.017	35.900	-3.351	35.900	-3.351	35.900	35.900	35.900	35.900	
3.284	-2.739	44.141	-2.989	44.141	-3.238	44.141	-3.488	44.141	-3.738	44.141	-3.738	44.141	44.141	44.141	44.141	
3.181	-3.718	61.090	-3.884	61.090	-4.050	61.090	-4.216	61.090	-4.381	61.090	-4.381	61.090	61.090	61.090	61.090	
3.086	-5.321	112.789	-5.403	112.789	-5.486	112.789	-5.569	112.789	-5.651	112.789	-5.651	112.789	112.789	112.789	112.789	
0.100																
23.379	5.914	66.510	2.682	66.510	-0.550	66.510	-3.781	66.510	-7.013	66.510	-7.013	66.510	66.510	66.510	66.510	
19.539	5.612	59.184	2.553	59.184	-0.505	59.184	-3.563	59.184	-6.621	59.184	-6.621	59.184	59.184	59.184	59.184	
16.311	5.303	52.846	2.418	52.846	-0.467	52.846	-3.352	52.846	-6.237	52.846	-6.237	52.846	52.846	52.846	52.846	
13.595	4.988	47.368	2.275	47.368	-0.437	47.368	-3.150	47.368	-5.862	47.368	-5.862	47.368	47.368	47.368	47.368	
11.310	4.664	42.638	2.124	42.638	-0.416	42.638	-2.956	42.638	-5.496	42.638	-5.496	42.638	42.638	42.638	42.638	
9.387	4.331	38.566	1.963	38.566	-0.405	38.566	-2.773	38.566	-5.141	38.566	-5.141	38.566	38.566	38.566	38.566	
7.768	3.988	35.073	1.791	35.073	-0.405	35.073	-2.602	35.073	-4.799	35.073	-4.799	35.073	35.073	35.073	35.073	
6.403	3.632	32.098	1.606	32.098	-0.419	32.098	-2.444	32.098	-4.470	32.098	-4.470	32.098	32.098	32.098	32.098	
5.254	3.261	29.589	1.406	29.589	-0.448	29.589	-2.303	29.589	-4.157	29.589	-4.157	29.589	29.589	29.589	29.589	
4.285	2.873	27.514	1.189	27.514	-0.495	27.514	-2.180	27.514	-3.864	27.514	-3.864	27.514	27.514	27.514	27.514	
3.467	2.463	25.850	0.949	25.850	-0.565	25.850	-2.079	25.850	-3.593	25.850	-3.593	25.850	25.850	25.850	25.850	
2.777	2.027	24.595	0.683	24.595	-0.662	24.595	-2.006	24.595	-3.350	24.595	-3.350	24.595	24.595	24.595	24.595	
2.195	1.557	23.788	0.382	23.788	-0.793	23.788	-1.968	23.788	-3.143	23.788	-3.143	23.788	23.788	23.788	23.788	
1.704	1.043	23.490	0.037	23.490	-0.969	23.490	-1.975	23.490	-2.982	23.490	-2.982	23.490	23.490	23.490	23.490	
1.289	0.468	23.865	-0.369	23.865	-1.207	23.865	-2.045	23.865	-2.882	23.865	-2.882	23.865	23.865	23.865	23.865	
0.938	-0.195	25.265	-0.864	25.265	-1.534	25.265	-2.233	25.265	-2.872	25.265	-2.872	25.265	25.265	25.265	25.265	
0.642	-0.998	28.539	-1.499	28.539	-2.001	28.539	-2.502	28.539	-3.044	28.539	-3.044	28.539	28.539	28.539	28.539	
0.391	-2.056	36.278	-2.390	36.278	-2.724	36.278	-3.058	36.278	-3.392	36.278	-3.392	36.278	36.278	36.278	36.278	
0.179	-3.738	61.505	-3.905	61.505	-4.072	61.505	-4.239	61.505	-4.406	61.505	-4.406	61.505	61.505	61.505	61.505	
u.vv																
3.324	2.331	25.894	0.828	25.894	-0.674	25.894	-2.177	25.894	-3.680	25.894	-3.680	25.894	25.894	25.894	25.894	
3.311	2.323	25.867	0.824	25.867	-0.676	25.867	-2.175	25.867	-3.670	25.867	-3.670	25.867	25.867	25.867	25.867	
3.297	2.315	25.840	0.819	25.840	-0.677	25.840	-2.174	25.840	-3.660	25.840	-3.660	25.840	25.840	25.840	25.840	
3.284	2.307	25.814	0.814	25.814	-0.679	25.814	-2.172	25.814	-3.655	25.814	-3.655	25.814	25.814	25.814	25.814	
3.271	2.299	25.788	0.810	25.788	-0.680	25.788	-2.170	25.788	-3.650	25.788	-3.650	25.788	25.788	25.788	25.788	
3.257	2.292	25.762	0.805	25.762	-0.682	25.762	-2.168	25.762	-3.645	25.762	-3.645	25.762	25.762	25.762	25.762	
3.244	2.284	25.736	0.800	25.736	-0.683	25.736	-2.167	25.736	-3.640	25.736	-3.640	25.736	25.736	25.736	25.736	
2.401	1.695	24.281	0.439	24.281	-0.817	24.281	-2.073	24.281	-3.299	24.281	-3.299	24.281	24.281	24.281	24.281	
2.299	1.609	24.144	0.384	24.144	-0.841	24.144	-2.065	24.144	-3.289	24.144	-3.289	24.144	24.144	24.144	24.144	
2.200	1.522	24.023	0.328	24.023	-0.866	24.023	-2.059	24.023	-3.279	24.023	-3.279	24.023	24.023	24.023	24.023	
2.104	1.433	23.919	0.271	23.919	-0.892	23.919	-2.054	23.919	-3.269	23.919	-3.269	23.919	23.919	23.919	23.919	
2.011	1.343	23.833	0.212	23.833	-0.920	23.833	-2.051	23.833	-3.259	23.833	-3.259	23.833	23.833	23.833	23.833	
1.922	1.252	23.766	0.151	23.766	-0.949	23.766	-2.049	23.766	-3.249	23.766	-3.249	23.766	23.766	23.766	23.766	
1.835	1.159	23.717	0.089	23.717	-0.980	23.717	-2.049	23.717	-3.239	23.717	-3.239	23.717	23.717	23.717	23.717	
1.503	-1.531	31.874	-1.949	31.874	-2.367	31.874	-2.786	31.874	-3.204	31.874	-3.204	31.874	31.874	31.874	31.874	
0.400	-2.021	35.936	-2.366	35.936	-2.711	35.936	-3.057	35.936	-3.402	35.936	-3.402	35.936	35.936	35.936	35.936	
0.304	-2.607	42.370	-2.880	42.370	-3.152	42.370	-3.425	42.370	-3.697	42.370	-3.697	42.370	42.370	42.370	42.370	
0.215	-3.349	53.695	-3.549	53.695	-3.749	53.695	-3.950	53.695	-4.150	53.695	-4.150	53.695	53.695	53.695	53.695	
0.133	-4.387	78.025	-4.515	78.025	-4.643	78.025	-4.772	78.025	-4.900	78.025	-4.900	78.025	78.025	78.025	78.025	

We note that no approximation has been made in the derivation of this expression. Before we change  $l$  to  $m$ , we make explicit the dependence of the absolute luminosity  $L(t)$  on  $\beta$ . Let us write in general

$$L(t) = L_0 \beta^{\epsilon} t^{\alpha}, \quad (7.8)$$

where the index  $\epsilon$  stands for evolution and will be discussed in the Appendices. At the same time,  $G(\beta)$  will be written as (see Paper I, discussion preceding eq. [2.44])

$$G(\beta) = G_0 \beta^{-\pi(G)} \equiv G_0 \beta^{-g}. \quad (7.9)$$

If one demands that  $G \sim 1/t$ , equations (4.1) and (7.9) imply that

$$g\epsilon = -1. \quad (7.10)$$

As discussed in the previous papers,  $\pi(G)$  is the power of  $G$  under scale transformations. With (7.8) and (7.9), we finally have from (7.7) ( $l = 10^{-2m/5} \times 2.52 \times 10^{-5}$  ergs  $\text{cm}^{-2} \text{s}^{-1}$ ;  $L = 10^{-2m/5} \times 3.02 \times 10^{35}$  ergs  $\text{s}^{-1}$ )

$$m = 5 \log F(z, \bar{q}_0) - \frac{5}{2}(e - g) \log \beta(t) + 5 \log (H_0/\bar{H}_0) + m_0, \quad (7.11)$$

$$m_0 = M - 97.45 + 5 \log (c/H_0) = M + 42.38 - 5 \log (H_0/100), \quad (7.12)$$

where  $M$  is the absolute magnitude, and  $H_0$  is measured in units of  $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$ .

In column (10) of Tables 1–4 we present the function  $F(z, \bar{q}_0)$ . In Tables 5–8 we present the values of  $m$  with  $m_0 = 0$  for four values of  $e$ . From equation (7.11) it is clear that the  $m$  versus  $z$  relation will yield  $\bar{q}_0$ , not  $q_0$ . Results are presented in Figures 5–8.

#### a) The Analytic Case: $k = 0$

Since in the  $k = 0$  case we have an explicit expression for  $R(t)$ , equation (5.3), we shall derive the  $m$  versus  $z$  relation directly and show that it is indeed a particular case of the general expression (7.7). To this end, we compute  $r_e$  directly using equation (5.3). We obtain after a short computation

$$r_e = \frac{r}{1 - r} \frac{c}{H_0 R_0} [1 - (1 + z)^{(r-1)/r}]. \quad (7.13)$$

Introducing equation (7.13) and the second of equations (5.6) into (7.4), we obtain

$$l = \frac{L(t)}{4\pi(c/H_0)^2} G(\beta) \left( \frac{\epsilon - 1}{\epsilon + 2} \right)^2 \frac{(1 + z)^{4(\epsilon-1)/(\epsilon+2)}}{[1 - (1 + z)^{(\epsilon-1)/(\epsilon+2)}]^2}. \quad (7.14)$$

On the other hand, from (7.6) and the first of (5.6), we have

$$F(z, \bar{q}_0 = \frac{1}{2}) = 2[1 + z - (1 + z)^{1/2}] = 2(1 + z)^{2(1-\epsilon)/(2+\epsilon)} [1 - (1 + z)^{(\epsilon-1)/(\epsilon+2)}]. \quad (7.15)$$

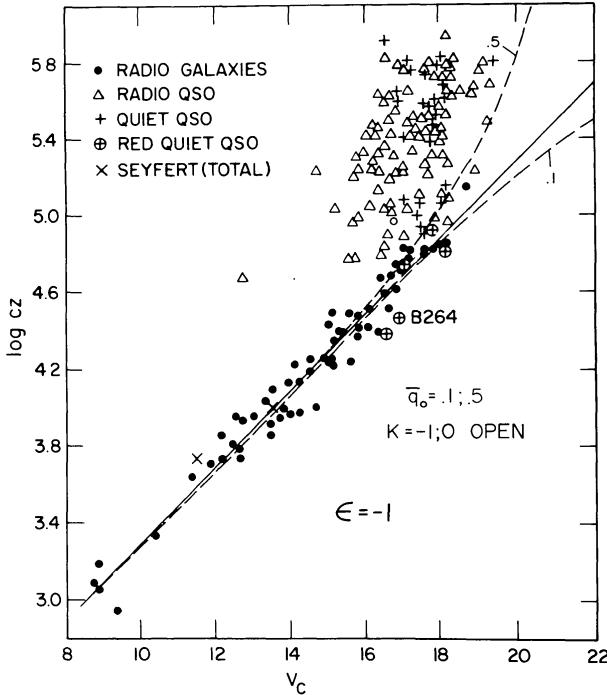


FIG. 5

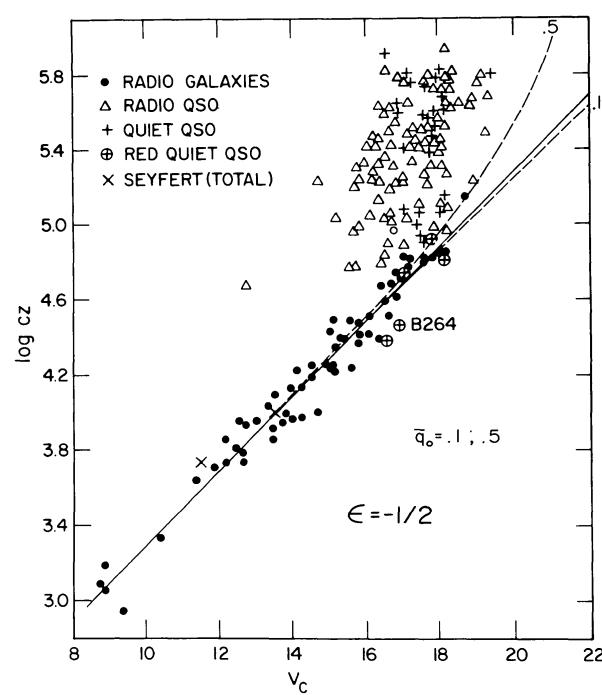


FIG. 6

FIG. 5.—The  $(m, z)$ -diagram, see eq. (7.11), for  $\epsilon = -1$  and two values of  $\bar{q}_0 = 0.1$  and  $\frac{1}{2}$ . The evolutionary parameter is  $e = -1.5$ . The value of the normalization  $m_0$  is 21.03 (the observational points are from Fig. 3 of Sandage 1972a).

FIG. 6.—Same as Fig. 5, for  $\epsilon = -\frac{1}{2}$ ;  $e = -1.7$

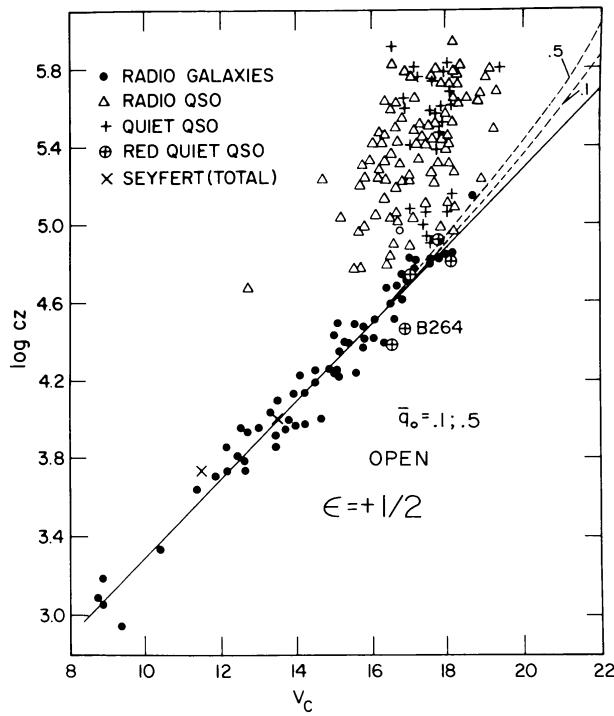


FIG. 7

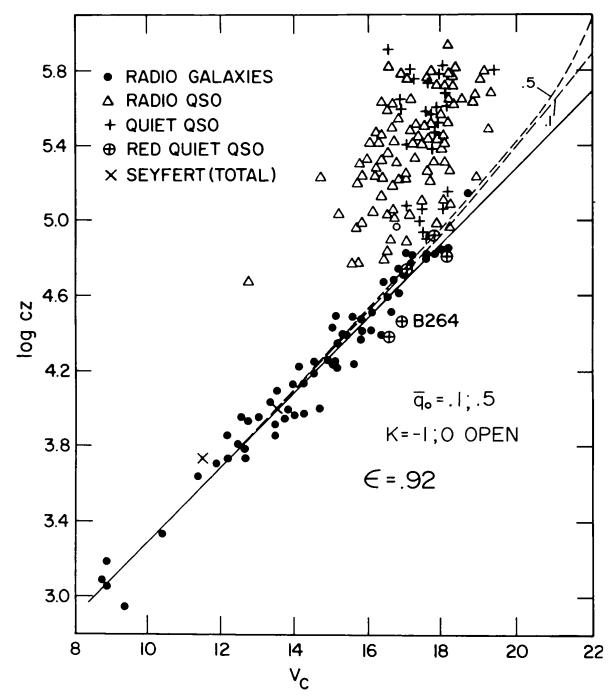


FIG. 8

FIG. 7.—Same as Fig. 5, for  $\epsilon = \frac{1}{2}$ ;  $e = -1.0$ FIG. 8.—Same as Fig. 5, for  $\epsilon = 0.92$ ;  $e = -1.2$ 

Use of equation (7.15) and the second of equations (5.4) shows that (7.7) is indeed identical to (7.14). In particular, from (7.11) and (7.15), we derive, making use of (5.4) and (5.6),

$$\epsilon = -1:$$

$$m = 5 \log z + 5 \log 2(2+z)(1+z)^2 + \frac{1}{2}(e-g) \log(1+z) + 5 \log \frac{1}{4} + m_0; \quad (7.16)$$

$$\epsilon = -\frac{1}{2}:$$

$$m = 5 \log z + \frac{5}{2}e \log(1+z) + 5(1 - \frac{1}{2}g) \log(1+z) + m_0; \quad (7.17)$$

$$\epsilon = +\frac{1}{2}:$$

$$m = 5 \log [(1+z)^{1/5} - 1] + \log(1+z) - \frac{3}{2}(e-g) \log(1+z) + 5 \log 5 + m_0. \quad (7.18)$$

Equations (7.16)–(7.18) for the case  $\epsilon g = -1$  (eqs. [7.9]–[7.10]) and  $m_0 = 0$  are tabulated in Tables 5–8.

### VIII. THE EFFECTIVE DECELERATION PARAMETER

Since the physically most important feature of the present theory consists of having disentangled atomic from gravitational times, it is to be expected that the higher the time derivative one considers, the more pronounced the effect will be.

We shall illustrate this phenomenon by studying the  $m$  versus  $z$  relation in the limit of small  $z$ 's, in which case the exact relation (7.7) can be rewritten in a form that depends on measurable quantities only.

At the same time, an effective curvature  $q_0^*$  will be defined which clearly displays the desired effect.

For small  $z$ 's, the function  $F(z, \bar{q}_0)$  (eq. [7.6]) can be written as

$$F(z, \bar{q}_0) \approx z[1 + \frac{1}{2}(1 - \bar{q}_0)z]. \quad (8.1)$$

Using (7.8), (7.9), (8.1), (5.13), and (5.14), equation (7.7) can be transformed to the familiar form

$$l = \frac{1}{4\pi} \left( \frac{H_0}{c} \right)^2 \frac{L(t_0)}{F^2(z, q^*)}, \quad (8.2)$$

where the effective curvature  $q^*$  is defined as

$$q^* = \bar{q}_0(1 + h_0/H_0)^2 - h_0/H_0 - (1 + Q_0)(h_0/H_0)^2 - (e - g)h_0/H_0. \quad (8.3)$$

*a) The Standard Case*

Equation (8.3) can be reduced to the standard case. To that end we put  $\epsilon \rightarrow 0$  ( $t_E \equiv \bar{t}$ ),  $g \rightarrow 0$ ,  $H_0 \rightarrow \bar{H}_0$ ,  $h_0 \rightarrow 0$ ,  $Q_0 \rightarrow 0$ , but  $e\epsilon = \eta$ . In fact,

$$L(t) \sim \beta^e \sim t^{-\epsilon e} \sim t_E^{-\eta}. \quad (8.4)$$

We obtain

$$q^*_{\text{s.c.}} = \bar{q}_0 + \frac{1}{\bar{H}_0 \bar{t}_0} \eta = \bar{q}_0 - \frac{1}{\bar{H}_0} \left( \frac{L'}{L} \right)_0, \quad (8.5)$$

a well-known expression in standard cosmology.

*b) Present Cosmology*

We shall rewrite (8.3) so as to exhibit a form that will help us compare it with the standard case. We shall write

$$q^* = \bar{q}_0 \left( \frac{\bar{H}_0}{H_0} \right)^2 + (g - 1) \frac{h_0}{H_0} - (1 + Q_0) \left( \frac{h_0}{H_0} \right)^2 + \frac{1}{H_0 t_0} \eta. \quad (8.6)$$

Contrary to (8.5), where  $\bar{q}_0$  is multiplied by unity, here we have a magnifying effect due to  $\bar{H}_0/H_0$  which makes the  $q^*$  versus  $\bar{q}_0$  even more nonlinear than usual. To this, we must add the two extra terms depending entirely on the first and second derivatives of  $\beta$ . For future use, we shall present (8.6) for  $k = 0$ ,  $\bar{q}_0 = \frac{1}{2}$ . Using (5.4), we obtain  $\epsilon = -1$ :

$$q^* = -1 + 3\eta; \quad (8.7a)$$

$\epsilon = -\frac{1}{2}$ :

$$q^* = 1 + 2\eta; \quad (8.7b)$$

$\epsilon = +\frac{1}{2}$ :

$$q^* = \frac{1}{3}(13 + 3e); \quad (8.7c)$$

$\epsilon = 1$ :

$$q^* = 3 + \eta. \quad (8.7d)$$

*c) The Value of  $\eta$* 

Using (A22), (A13), and (A30), we obtain

$$\begin{aligned} \epsilon e \equiv \eta &= \underbrace{\epsilon(g-1)x + \frac{\alpha-x}{\alpha-1}(1+\epsilon g \delta) - \epsilon g \delta + \epsilon \gamma \frac{1-x}{1-\alpha} + \frac{R}{R+1}}_{(\text{evol.})} \\ &\times \left[ \epsilon(g-1) + 1 + \epsilon g \delta + \frac{\gamma \epsilon}{\alpha-1} + (1+\epsilon g \delta) \frac{\alpha}{1-\alpha} \right] + \underbrace{\epsilon(g-1) - \frac{1}{2} \frac{1}{1-t_c/t_0}}_{(\text{dyn. fr.})}. \end{aligned} \quad (8.8)$$

When  $\epsilon \rightarrow 0$ ,  $g \rightarrow 0$ , we obtain

$$\eta = \underbrace{\frac{\alpha-x}{\alpha-1} + \frac{R}{R+1} \frac{1}{1-\alpha}}_{(\text{evol.})} - \underbrace{\frac{1}{2} \frac{1}{1-t_c/\bar{t}_0}}_{(\text{dyn. fr.})} \quad (8.9)$$

a well-known expression recently discussed by Gunn and Tinsley (1976).

*d) The Complete Solution*

Even though we have derived an expression for  $\eta = e\epsilon$ , it would not be convenient to use it at this stage since we want to stress general properties of our cosmology. We shall therefore solve equation (8.6) for different values of  $\eta$ . The resulting  $q^*$  versus  $\bar{q}_0$  curves are shown in Figures 9–12 together with the ones corresponding to (8.5). Examination of (8.3) shows that, in the present cosmology, in addition to the intrinsic luminosity variation there is also the evolution due to the relative variation of gravitational and atomic force strengths (whose effects can also be seen from eq. [7.4b]). This is expected: *by studying the gravitational evolution of the universe through atomic*

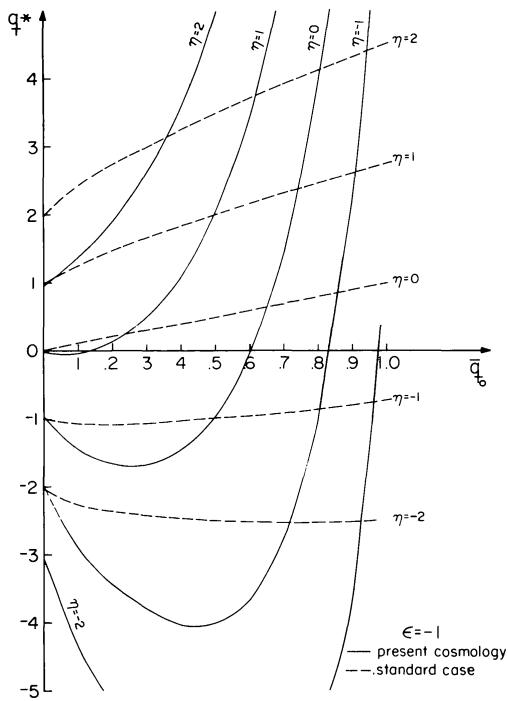


FIG. 9

FIG. 9.—The effective deceleration parameter  $q^*$  as determined from the  $(m, z)$ -relation, eq. (8.2), as a function of the geometrical  $\bar{q}_0$ , eq. (8.3), for different values of the evolutionary parameter  $\eta$ , eq. (8.4);  $\epsilon = -1$ .

FIG. 10.—Same as Fig. 9 for  $\epsilon = -\frac{1}{2}$

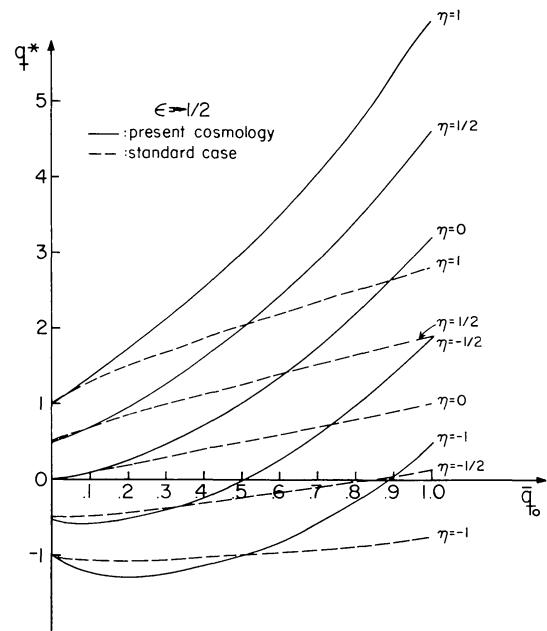


FIG. 10

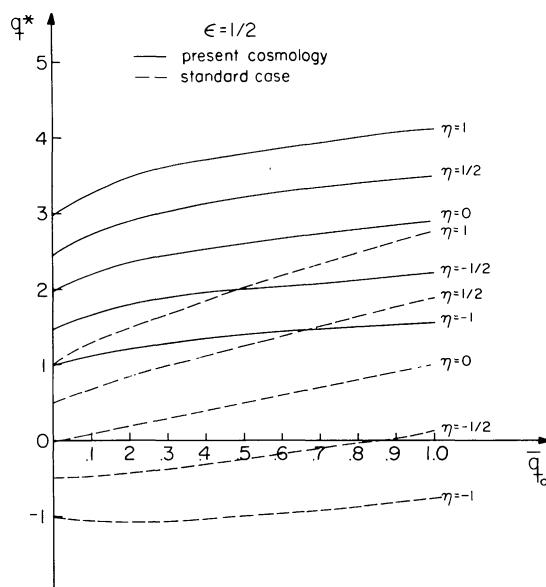


FIG. 11

FIG. 11.—Same as Fig. 9 for  $\epsilon = +\frac{1}{2}$

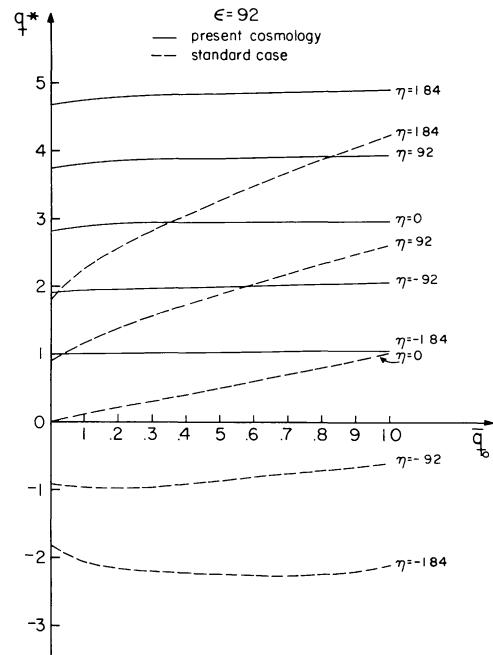


FIG. 12

FIG. 12.—Same as Fig. 9 for  $\epsilon = 0.92$

instruments, we have no *a priori* reason to assume we can see clearly the gravitational dynamics, in this case characterized by the parameter  $\bar{q}_0$ . Instead, we should allow for distortions as if looking through a lens, which may be magnifying or reducing depending on the evolution of the scaling function. This is most easily seen from the first term on the right-hand side of (8.3). If  $\beta$  is an increasing function of time, then  $h_0$  is positive and the multiplicative factor  $(1 + h_0/H_0)^2$  is greater than unity. In this case, atomic instruments amplify small differences in the geometrical structure, given by  $\Delta\bar{q}_0$ . The reverse is true if  $\beta$  is a decreasing function of time. Of course, when  $\beta$  is prescribed as a function of the cosmic times,  $h_0$  becomes a function of the age of the universe, which in turn is an implicit function of  $\bar{q}_0$ . The "lens effect" thus becomes more complex. This "lens effect" can be seen from Figures 9–12 by comparing the slopes of the full curves with those of standard cosmology (dashed lines). For a given observational uncertainty in  $q^*$ , a steep slope reduces the uncertainty in  $\bar{q}_0$ .

For positive  $\epsilon$ , the solid curves generally have slopes smaller than those of standard cosmology. Such positive or negative "magnification" of the gravitational dynamical effects through the atomic lens can also be seen in Figures 5–8 and 13–20, where curves corresponding to  $\bar{q}_0 = 0.1$  and 0.5 are presented.

In each case, the separation between the two curves is enhanced if  $\epsilon < 0$  and reduced if  $\epsilon > 0$ .

It is also of interest to see from Figures 9–12 how an uncertainty in  $\eta$  affects the determination of  $\bar{q}_0$ , given an observed  $q^*$ .

Here again, for a family of more or less horizontal curves, a small shift in  $\eta$  causes a large shift in  $\bar{q}_0$ . This has been the situation in standard cosmology, and it presents an obstacle preventing the determination of the true

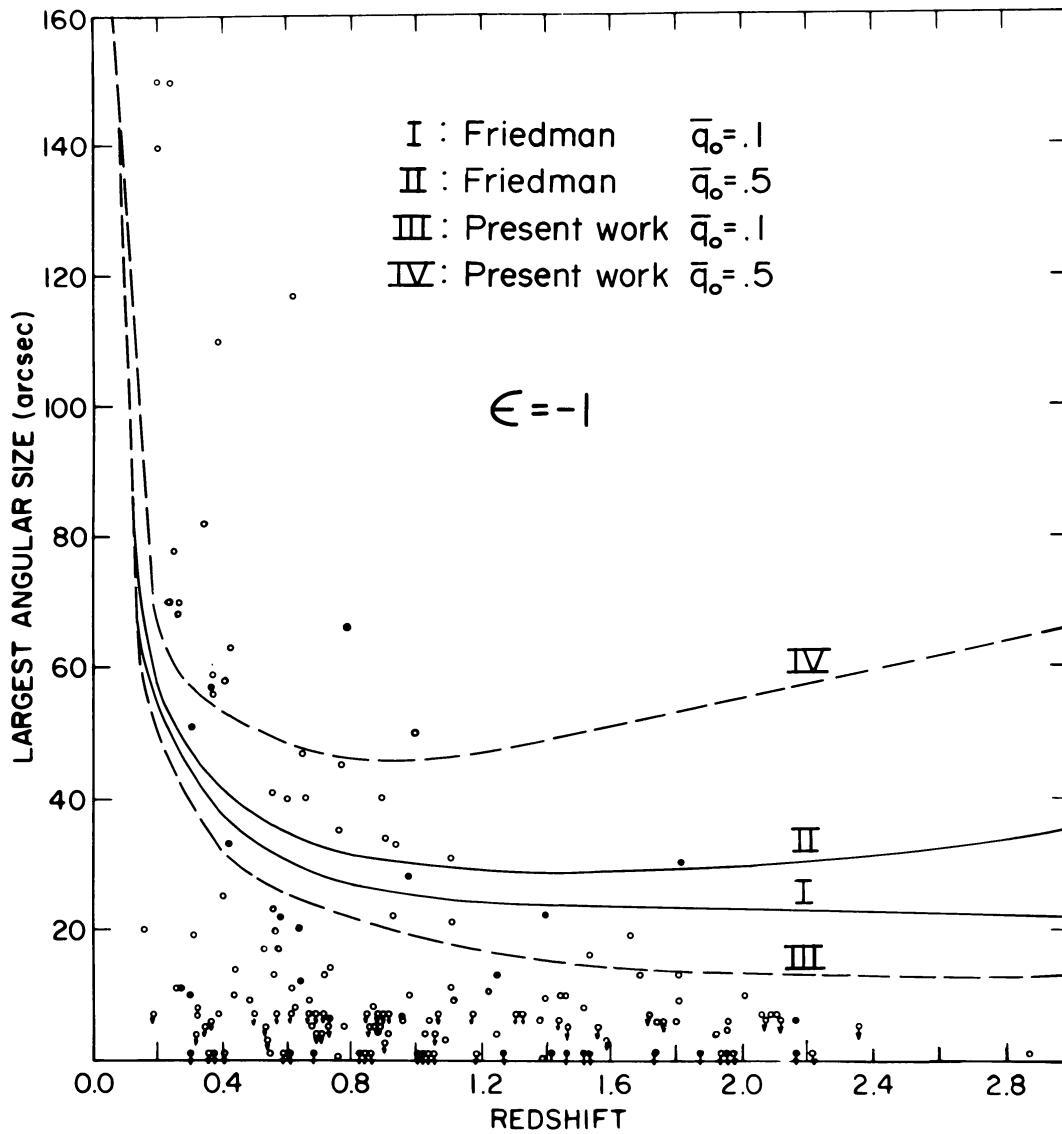


FIG. 13.—The metric angular diameters versus  $z$ , eq. (9.2), for two values of  $\bar{q}_0 = 0.1$  and 0.5 (dashed lines). Predictions of standard cosmology are represented by full lines;  $\epsilon = -1$ . (The observational points are from Wardle and Miley 1974.)

curvature of space. For positive  $\epsilon$ , the situation has not improved. For negative  $\epsilon$ , on the other hand, there are regions where a change in  $\eta$  produces relatively unimportant changes in  $\bar{q}_0$ .

#### IX. METRIC ANGULAR DIAMETERS

Consider two events at the points  $A(r_e, \theta_e, \varphi_e)$  and  $B(r_e, \theta_e + \Delta\theta_e, \varphi_e)$  occurring at the same time  $t_e$ , and let the observer be located at  $(0, 0, 0)$  at the time  $t_0$ . The two emission events are separated by a local distance  $t$ .

The metric angular diameter  $\theta_m$  is defined as

$$\theta_m = \frac{y}{r_e R(t_e)} . \quad (9.1)$$

Using equations (7.5), (5.1), and (3.3), equation (9.1) becomes

$$\theta_m = \theta_0 \left( \frac{H_0 t_0 - \epsilon}{H_0 t_0} \right) \frac{\beta(t)(1+z)^2}{F(z, \bar{q}_0)} , \quad (9.2)$$

where

$$\theta_0 = 8.6 \left( \frac{y}{250 \text{ kpc}} \right) \left( \frac{H_0}{50} \right) \text{ arcsec} . \quad (9.3)$$

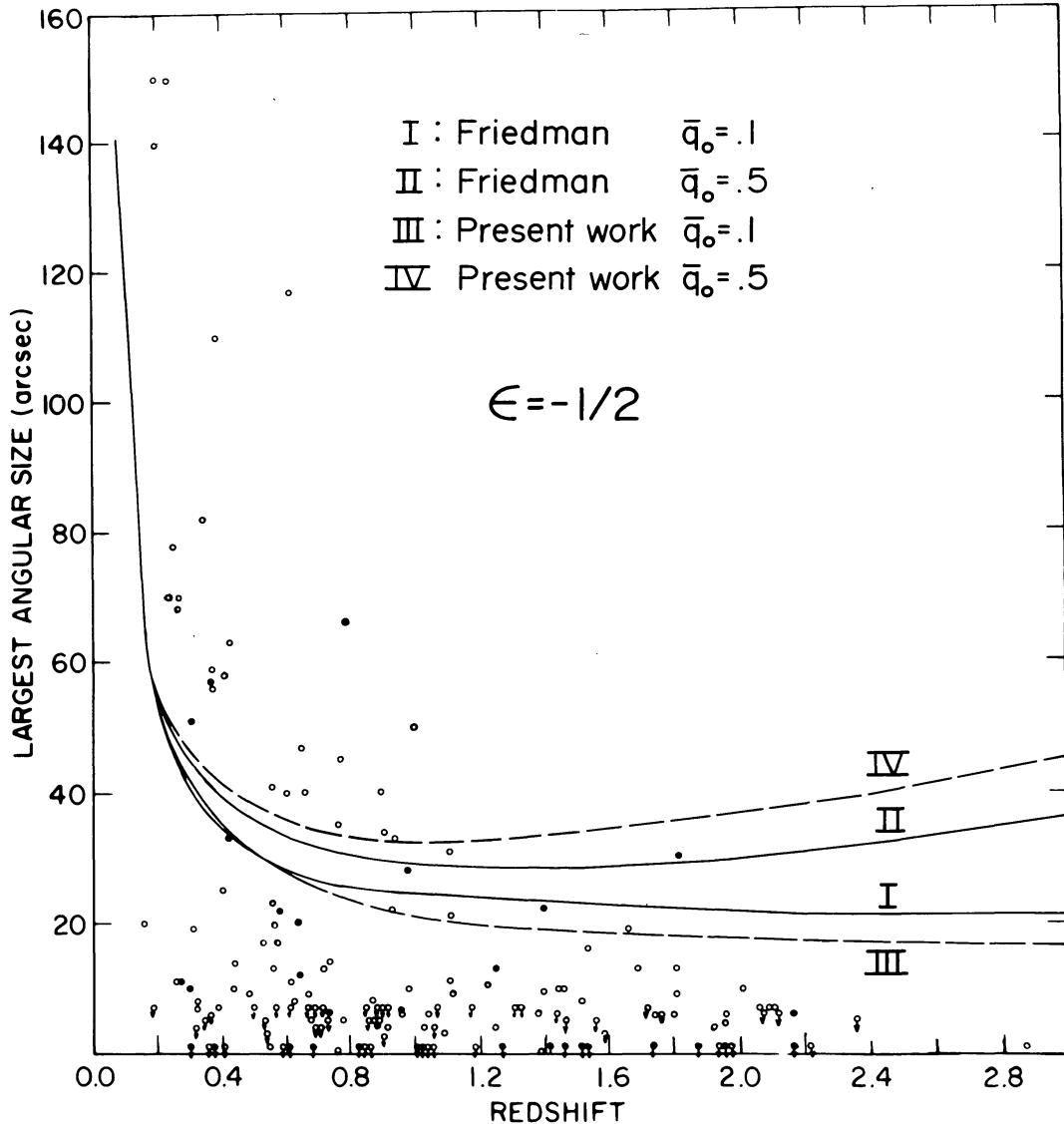
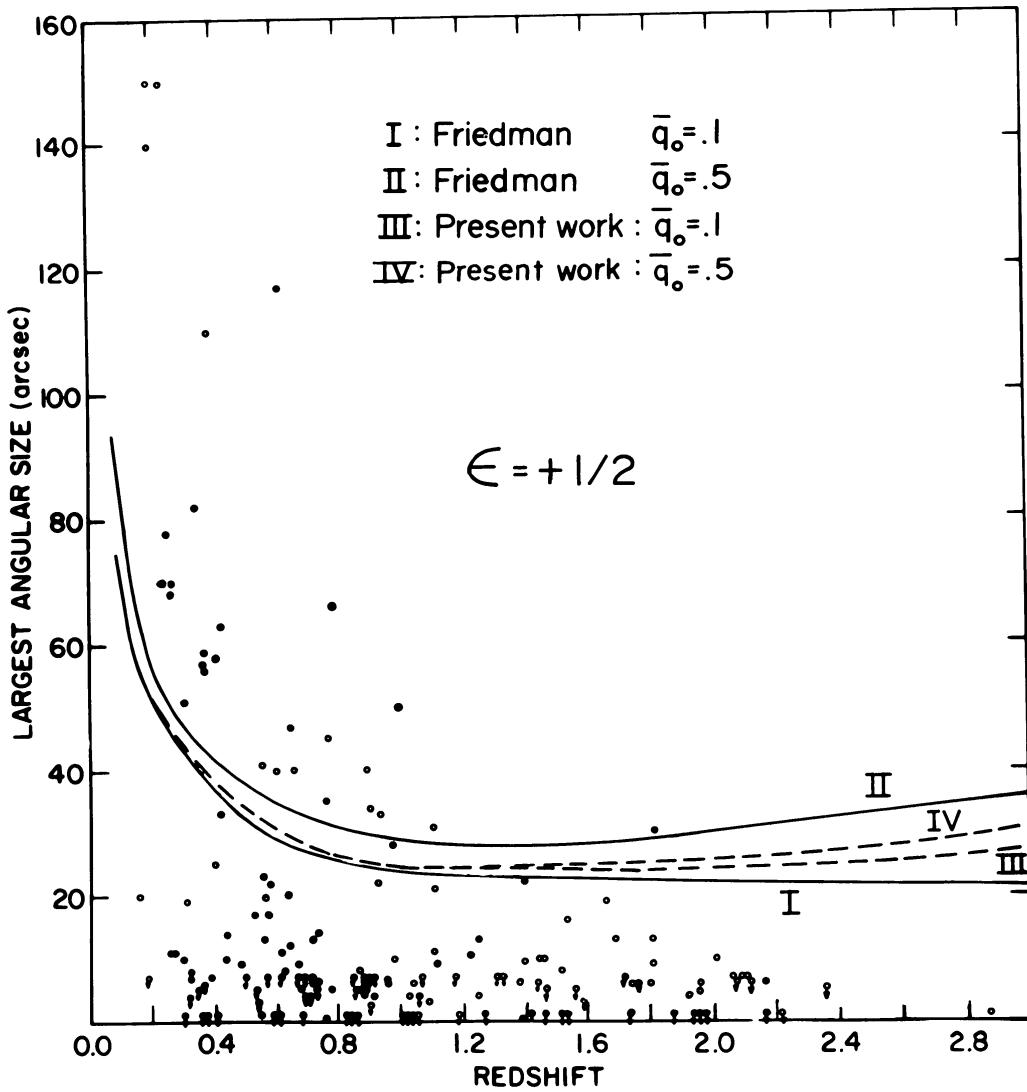


FIG. 14.—Same as Fig. 13 for  $\epsilon = -\frac{1}{2}$

FIG. 15.—Same as Fig. 13 for  $\epsilon = \frac{1}{2}$ 

Using the values of  $z$ ,  $\beta$ , and  $F(z, \bar{q}_0)$  versus  $z$  given in Tables 1–4,  $\theta_m$  can be computed from (9.2). The numerical values quoted in Tables 5–8 correspond to  $y = 250$  kpc and  $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ . We have not included a possible  $z$ -dependence of the radius  $y$ . A scaling of the form  $y(z) = y(0)(1+z)^n$  can be easily superposed on the values quoted in the tables once a particular model has been chosen. The results are presented in Figures 13–16.

#### a) The Analytic Case: $k = 0$

Using (5.4), (5.6), and (7.15),  $\theta_m$  can be expressed as

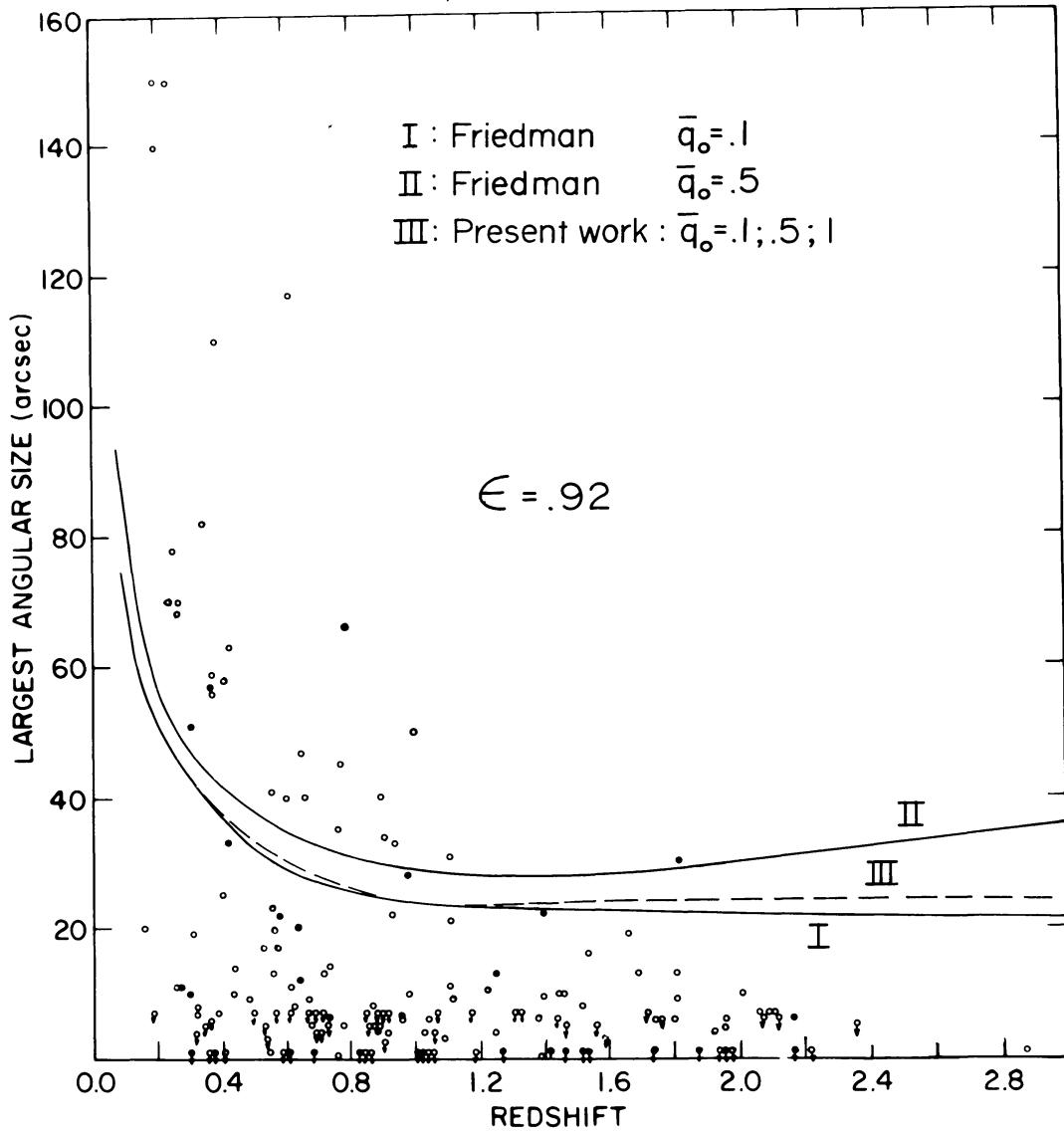
$$\theta_m = \theta_0 \left( \frac{1-\epsilon}{2+\epsilon} \right) \frac{1+z}{1-(1+z)^{(\epsilon-1)/(\epsilon+2)}}. \quad (9.4)$$

For  $\epsilon = 0$ , (9.4) reduces to the well-known result in ordinary cosmology corresponding to the Einstein-de Sitter universe. Equation (9.4) can be compared with the values quoted in Tables 5–8.

#### X. ISOPHOTAL ANGLES

In order to derive the isophotal angular diameters  $\theta_i$ , we shall first define the surface brightness  $B$  as

$$B = I/\theta_m^2, \quad (10.1)$$

FIG. 16.—Same as Fig. 13 for  $\epsilon = 0.92$ 

where  $l$  is given by (7.4). Using (9.1), we now have

$$B \sim (1+z)^{-4} L(t) \beta^2 G(\beta) y^{-2}. \quad (10.2)$$

The determination of  $\theta_i$  is usually made adopting a formula (due to Hubble) giving the variation of  $B$  versus  $\theta$ , namely,

$$\theta_i/\theta_m \sim B^{1/p} \quad (p \approx 2). \quad (10.3)$$

We then have

$$\frac{\theta_i}{\theta_m} \sim (1+z)^{-4/p} \left[ \frac{L(t) \beta^2 G(\beta)}{y^2} \right]^{1/p}, \quad (10.4)$$

and using (9.1) (with  $p = 2$ )

$$\theta_i \approx \left[ \frac{L(t) \beta^2 G(\beta)}{r_e^2 (1+z)^2} \right]^{1/2}. \quad (10.5)$$

Comparing (10.5) with (7.4), we obtain the final result

$$\log \theta_i = -\frac{1}{5} m + \text{const.} \quad (10.6)$$

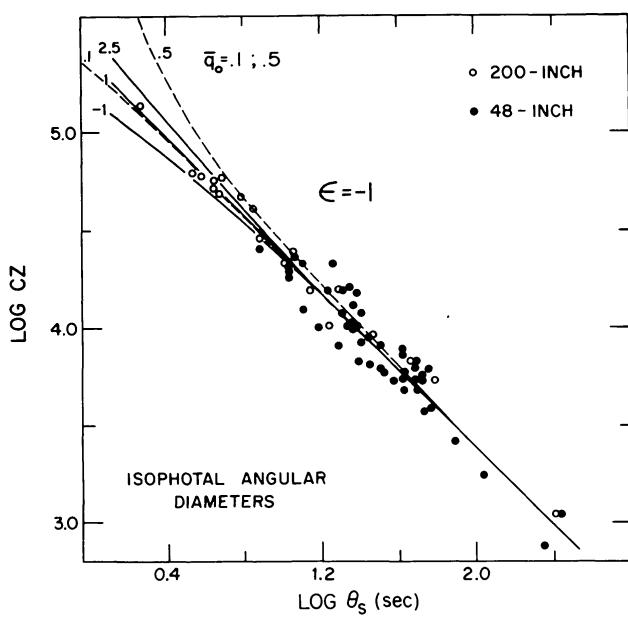


FIG. 17

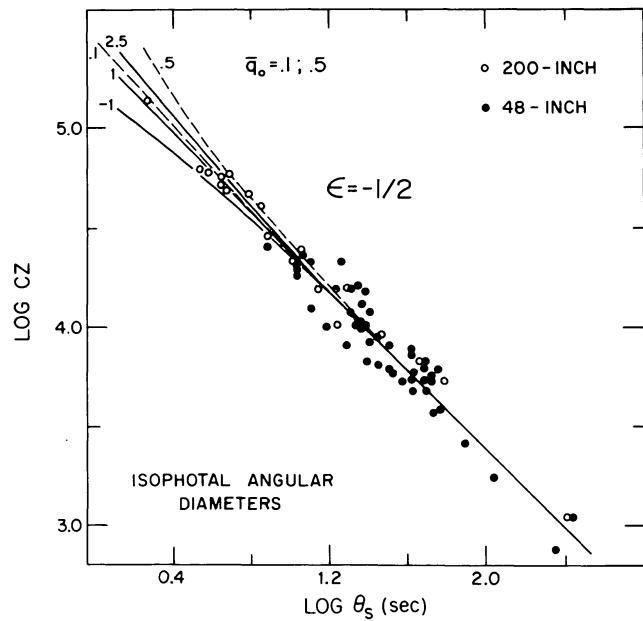


FIG. 18

FIG. 17.—Isophotal angular diameter  $\theta_i$  versus  $z$ , eq. (10.6), for two values of  $\bar{q}_0 = 0.1$  and  $0.5$ . The value of  $e$  is the same as for Fig. 5:  $e = -1$ . (The observational points are from Fig. 5 of Sandage 1972b.)

FIG. 18.—Same as Fig. 17 for  $e = -\frac{1}{2}$  (value of  $e$  from Fig. 6)

The values of  $\log \theta_i$  are presented in Figures 17–20.

#### XI. DISTANCES

From (7.4) and (9.1) we can derive  $d_L$ , the luminosity distance, as well as  $d_A$ , the angular diameter distance, as

$$d_L = \frac{r_e}{R(t_e)\beta^{(2+e)/2}G^{1/2}}, \quad d_A = r_e R(t_e), \quad (11.1)$$

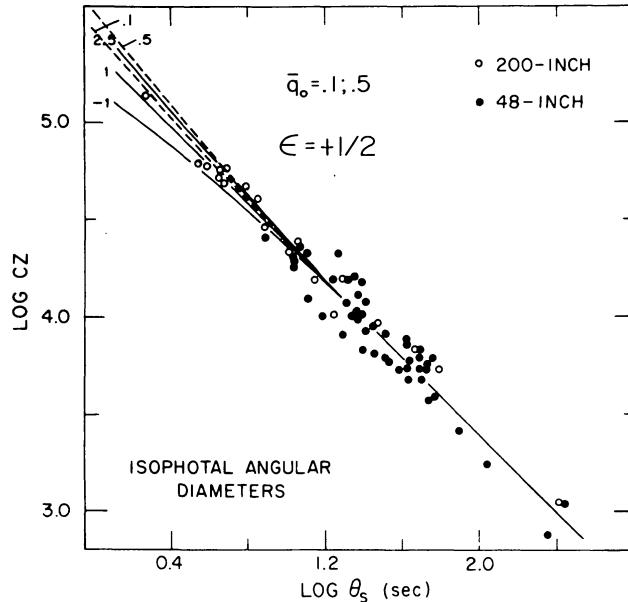


FIG. 19

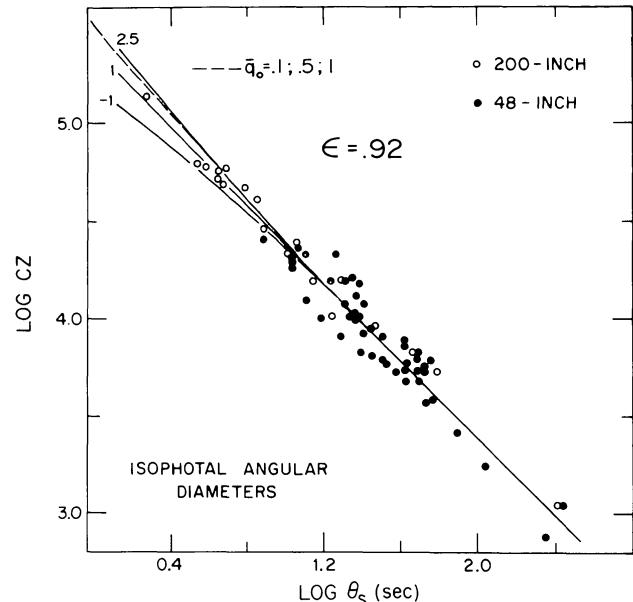


FIG. 20

FIG. 19.—Same as Fig. 17 for  $e = \frac{1}{2}$  (value of  $e$  from Fig. 7)

FIG. 20.—Same as Fig. 17 for  $e = 0.92$  (value of  $e$  from Fig. 8)

so that

$$\frac{d_A}{d_L} = \frac{1}{(1+z)^2} \beta^{(2+\epsilon)/2} G^{1/2}(\beta). \quad (11.2)$$

Using equation (5.6) for  $\beta$  in the  $k = 0$ , we obtain

$\epsilon = -1$  ( $g = 1$ ):

$$\frac{d_A}{d_L} = \frac{1}{(1+z)^2} (1+z)^{-3(1+\epsilon)/2}; \quad (11.3)$$

$\epsilon = -\frac{1}{2}$  ( $g = 2$ ):

$$\frac{d_A}{d_L} = \frac{1}{(1+z)^2} (1+z)^{-\epsilon/2}; \quad (11.4)$$

$\epsilon = +\frac{1}{2}$  ( $g = -2$ ):

$$\frac{d_A}{d_L} = \frac{1}{(1+z)^2} (1+z)^{3(2+\epsilon/2)/5}; \quad (11.5)$$

$\epsilon = 0.92$  ( $g = -1.087$ ):

$$\frac{d_A}{d_L} = \frac{1}{(1+z)^2} (1+z)^{1.459+0.4726\epsilon}. \quad (11.6)$$

Another way of expressing the experimental results is to say that the luminosity distance increases slowly with  $z$  (and even decreases after a maximum) and the angular distance increases quickly with  $z$ . This means that a cosmology in which  $d_A/d_L$  is a strong function of  $z$  is a good candidate for the tests. We have, using equations (11.2), (7.9), (5.1), and (3.3),

$$\frac{d_A}{d_L} = (1+z)^{-2+[2(2+\epsilon)\epsilon+1]/(2t_0H_0)}, \quad (11.7)$$

which is only an approximation for  $z \sim 0$ . For  $\bar{q}_0 = 0$  or  $\frac{1}{2}$ , equation (11.7) is exact. We want the exponent to be as large as possible and positive.

## XII. THE $[N(m), m]$ -RELATION

A test that is becoming increasingly important is the number count, i.e., the number of sources (per square degree) brighter than apparent magnitude  $m$ . The most widely used form of this test is in reference to optical quasars.

A formula of the type

$$\log N(m) = a + bm \quad (12.1)$$

is often employed for its simplicity: the discussion then concentrates on the value of the parameter  $b$  predicted by the theory versus the one obtained by fitting (12.1) to the data.

For a Euclidean space  $b = 0.6$ ; all Friedman universes with  $\Lambda = 0$  predict a value of  $b$  less than 0.6. However, evidence has been repeatedly adduced for values in excess of 0.6: Braccesi and Formiggini (1969) have concluded that  $b = 0.72$ ; Sandage and Luyten (1969) suggested  $b = 0.75$ , although Setti and Woltjer (1973) have concluded that all the data until 1973 were actually consistent with  $b = 0.6$ . In the most recent work on the subject, Green and Schmidt (1978) have presented what they believe to be a clear indication of a value of  $b$  much larger than any of the ones proposed so far; in fact, their analysis indicates  $b = 0.93$ , a value impossible to reconcile with any purely geometrical cosmology. Strong evolutionary trends in the number of quasars are then invoked as the logical implication of such results.

In this section we shall derive the  $N(m)$  versus  $m$  relation pertaining to the present cosmology and we shall then discuss the corresponding values of  $b$ .

To that end, let us first derive  $N(z)$ , the number of sources as a function of redshift. If we start with a Robertson-Walker metric (in atomic units)

$$ds^2 = dt^2 - R^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) \quad (12.2)$$

by virtue of the transformation

$$du = \frac{dr}{(1-kr^2)^{1/2}}, \quad u = (-k)^{-1/2} \sinh^{-1} [(-k)^{1/2}r], \quad (12.3)$$

we obtain

$$ds^2 = dt^2 - R^2(t)[du^2 + \Sigma^2(u)d\Omega^2], \quad (12.4)$$

where

$$r = \Sigma(u) = (-k)^{-1/2} \sinh [(-k)^{1/2}u]. \quad (12.5)$$

The volume element computed from (12.1) is

$$dv = g^{1/2}drd\theta d\phi = R^3(t)(1 - kr^2)^{-1/2}r^2dr \sin \theta d\theta d\phi; \quad (12.6)$$

consequently the total number of sources is given by

$$\mathcal{N} = 4\pi n_0 R_0^3 \int_0^u du \Sigma^2(u), \quad [n(t)R^3(t) = \text{const.}] \quad (12.7)$$

The number of sources per square degree is then

$$N = \mathcal{N}/\mathcal{Q}, \quad (12.8)$$

where  $\mathcal{Q}$  is the total number of square degrees in the sky. Integration of (12.7) yields

$$N = \frac{1}{4}N_0(-k)^{-3/2}(\sinh 2y - 2y), \quad (12.9)$$

where

$$N_0 = 4\pi n_0 R_0^3/\mathcal{Q}; \quad y = (-k)^{1/2}u. \quad (12.10)$$

Using (12.3), (7.5), and (2.5) to express  $r$  in terms of  $\bar{q}_0$  and  $z$ , we finally obtain

$$y = \sinh^{-1} \left[ (1 - 2\bar{q}_0)^{1/2} \frac{F(z, \bar{q}_0)}{1+z} \right]. \quad (12.11)$$

Equations (12.9) and (12.11) solve the problem of finding  $N$  versus  $z$ , since the relation between  $z$  and  $x$  has already been given in Tables 1–4. Equations (12.9) and (12.11) are formally identical to the ones in standard cosmology: however, in the present context  $z$  is not to be identified with the observed redshift, which is the atomic one,  $z$ . The  $N(z)$  versus  $z$  relation is therefore different in the present context than in standard cosmology.

Explicitly, we have

$k = 0$ :

$$N = \frac{1}{3}N_0 u^3 = \frac{1}{3} \frac{N_0 c^3}{\bar{H}_0^3 R_0^3} \left[ \frac{F(z, \frac{1}{2})}{1+z} \right]^3; \quad (12.12)$$

$k = -1$ :

$$N = \frac{1}{4}N_0(\sinh 2y - 2y); \quad y = \sinh^{-1} \left[ (1 - 2\bar{q}_0)^{1/2} \frac{F(z, \bar{q}_0)}{1+z} \right]; \quad (12.13)$$

$k = +1$ :

$$N = \frac{1}{4}N_0(2y - \sin 2y); \quad y = \sin^{-1} \left[ (2\bar{q}_0 - 1)^{1/2} \frac{F(z, \bar{q}_0)}{1+z} \right]. \quad (12.14)$$

Let us now use (7.11) to eliminate  $z$  in favor of  $m$ . We have

$$\text{dex } [0.2(m - m_0)] = F(z, \bar{q}_0) \beta^{(g-e)/2} (H_0/\bar{H}_0). \quad (12.15)$$

#### a) Small Look-back Times

In this case, equations (12.9) and (12.11) reduce to

$$\mathcal{N} = \frac{4\pi}{3} \frac{n_0 c^3}{\bar{H}_0^3} \left( \frac{H_0}{\bar{H}_0} \right)^3 \left[ \frac{F(\bar{q}_0, z)}{1+z} \right]^3, \quad (12.16)$$

where, using (8.1),

$$\frac{F(\bar{q}_0, z)}{1+z} = z[1 - \frac{1}{2}(1 + \bar{q}_0)z]. \quad (12.17)$$

Finally using (5.15), we obtain

$$\mathcal{N} = \frac{4\pi}{3} \frac{n_0 c^3}{H_0^3} z^3 [1 - 3\Lambda z], \quad (12.18)$$

where

$$\Lambda \equiv \frac{1}{2}(1 + \bar{q}_0) \bar{H}_0/H_0 - \frac{\epsilon}{2H_0 t_0} \left( q_0 + \frac{\epsilon - 1}{t_0 H_0} \right) H_0/\bar{H}_0. \quad (12.19)$$

Let us now use  $l$  as derived in (8.2), i.e.  $[L_0 \equiv L(t_0)]$ ,

$$\left( \frac{H_0}{c} \right) \left( \frac{L_0}{4\pi l} \right)^{1/2} \equiv \varpi = z[1 + \frac{1}{2}(1 - q^*)z], \quad (12.20)$$

where we have again made use of (8.1) to express  $F(z, q^*)$  for small  $z$ . Eliminating  $z$  between (12.20) and (12.18), we obtain

$$\mathcal{N} = \frac{4\pi}{3} n_0 \left( \frac{L_0}{4\pi l} \right)^{3/2} \left\{ 1 - \frac{3}{2}\varpi [1 - q^* + (1 + \bar{q}_0) \bar{H}_0/H_0 - 2B(H_0/\bar{H}_0)] \right\}, \quad (12.21)$$

where

$$B \equiv \frac{\epsilon}{2H_0 t_0} \left( q_0 + \frac{\epsilon - 1}{t_0 H_0} \right).$$

Contrary to the well-known case in standard cosmology, the correction to the Euclidean case does depend on the curvature  $\bar{q}_0$ . In the case of standard cosmology,  $\epsilon \rightarrow 0$ ,  $g \rightarrow 0$ ,  $H_0 \rightarrow \bar{H}_0$ , we recover from (12.21), using (8.5),

$$\mathcal{N} = \frac{4\pi}{3} n_0 \left( \frac{L_0}{4\pi l} \right)^{3/2} \left\{ 1 - \frac{3\bar{H}_0}{c} \left( \frac{L_0}{4\pi l} \right)^{1/2} \left[ 1 + \frac{1}{2\bar{H}_0} \left( \frac{L'}{L} \right)_0 \right] \right\}, \quad (12.22)$$

a well-known expression in standard cosmology. Equation (12.21) can be further simplified. In fact, using expressions (2.5) and (8.3), one can show that the quantity in square brackets in (12.21) is equal to

$$2 - \frac{\epsilon(2 + e - g)}{H_0 t_0},$$

so that finally we have

$$\mathcal{N} = \frac{4\pi}{3} n_0 \left( \frac{L_0}{4\pi l} \right)^{3/2} \left\{ 1 - 3\varpi \left[ 1 - \frac{\epsilon(2 + e - g)}{2H_0 t_0} \right] \right\}. \quad (12.23)$$

This expression clearly indicates that a slope greater than the Euclidean one can be obtained as long as the value in the square brackets is negative.

### XIII. THE SLOPE OF THE $[\log N(m), m]$ -CURVE FOR QSOs AND THEIR $(m, z)$ -RELATION

Since evolutionary effects are very important, one cannot derive any reliable  $m$  versus  $z$  relation for QSOs without them.

However, because the physics of QSOs is still very poorly known, the evaluation of evolutionary effects is very uncertain since it is too model dependent.

For the case of elliptical galaxies we computed  $e$  (see eq. [7.8]), using the best model presently available, and the results shown in Figures 5–8 are satisfactory. However, there is no compelling reason why such values of  $e$  should also apply to QSOs in general, and the dotted curves in the figures should not be extrapolated much further beyond the region where they belong.

This leaves us with the problem of estimating  $e$  for QSOs. We proceed as follows. We demand that  $e$  satisfy the  $\log N(m)$  versus  $m$  relation with a given slope  $b$  determined by observations. For purpose of illustration we shall take  $b = 0.75$ , an average value among the ones quoted before. Once  $e$  is so determined, we shall use it to construct the  $m$  versus  $z$  relation for QSOs.

To understand the results better, we shall apply such a procedure first to the case of standard cosmology and then to our case. We shall show that within our cosmology the resultant  $m$  versus  $z$  relation gives a better fit than the one of standard cosmology.

Instead of presenting a full numerical solution to this problem, we shall employ the  $k = 0$  case since we can work out the problem entirely analytically. Using (5.6), equation (12.15) becomes

$$\text{dex } [0.2(m - m_0)] = F(z, \frac{1}{2})(1 + z)^p (H_0/\bar{H}_0), \quad (13.1)$$

TABLE 9  
VALUES OF THE PARAMETER  $\eta$  (eq. [13.8]) THAT YIELD  
A SLOPE OF 0.75 IN THE  $[\log N(m), m]$ -RELATION

$z$	$\eta$
2.....	1.51
2.5.....	1.48
3.....	1.46
3.5.....	1.45

where

$$p \equiv \frac{3}{4} \frac{\epsilon(g - e)}{1 - \epsilon}, \quad (13.2)$$

so that finally from (12.12) we obtain

$$N(m) = \frac{1}{3} N_0 \left( \frac{c}{H_0 R_0} \right)^3 \text{dex} [0.6(m - m_0)] (1 + z)^{-3-3p} \quad (13.3)$$

or

$$\log N(m) = 0.6m - 3(p + 1) \log(1 + z) + \text{const.} \quad (13.4)$$

The slope of the  $\log N(m)$  versus  $m$  curve can then be derived to be

$$\text{slope} \equiv \frac{d \ln N}{dm} = \frac{0.6}{1 - 2(p + 1) + 2(p + 1)(1 + z)^{1/2}}. \quad (13.5)$$

#### a) Standard Cosmology

In order to recover the expression pertaining to standard cosmology, we must (a) identify  $z$  with the observed redshift, and (b) put  $g \rightarrow 0$ ,  $\epsilon \rightarrow 0$ , i.e.,  $\bar{H}_0 \rightarrow H_0$  but keep the product  $\epsilon e$  finite, say  $\eta$ , so that (see [8.4])

$$L(t) \sim \beta^e \sim t^{-\epsilon e} \sim t_E^{-\eta} \sim (1 + z)^{3\eta/2}. \quad (13.6)$$

The value of  $p$  becomes

$$p = -\frac{3}{4}\eta. \quad (13.7)$$

Equation (13.5) remains unaltered with  $p$  given by (13.7); (13.1) becomes now

$$m = m_0 + 5 \log F(z, \frac{1}{2}) - \frac{15}{4}\eta \log(1 + z) \quad (13.8)$$

with

$$F(z, \frac{1}{2}) = 2[1 + z - (1 + z)^{1/2}].$$

Let us now look for a value of  $\eta$  such that the slope (13.5) is 0.75, a value intermediate among the ones quoted above. We obtain the results shown in Table 9, which, upon insertion in (13.8), yield the results shown in Table 10. Since for small  $z$ 's equation (13.8) becomes identical to equation (4) of Sandage (1972c), where  $m_0$  has been fitted to be 20.625, the magnitude  $m$  without  $\eta$  would be in the range 23–24.5, too high a value to fit even the envelope of the QSOs, as clear from Figures 5–8. The addition of the evolutionary term  $\eta$ , determined for the slope of the  $\log N(m)$  versus  $m$  relation, helps a great deal since it reduces column (2) of Table 10 by about 3 magnitudes. However, the final result (last column of Table 10) is still too high to fit the data.

TABLE 10  
THE  $(m, z)$ -RELATION OF STANDARD COSMOLOGY ( $k = 0$ ) USING  
THE VALUES OF  $\eta$  DETERMINED IN TABLE 9

$z$	$m - m_0$ ( $\eta = 0$ )	$m - m_0$ ( $\eta$ as for Table 9)
2.....	2.02	-0.68
2.5.....	2.56	-0.45
3.....	3.01	-0.28
3.5.....	3.38	-0.16

TABLE 11

VALUES OF THE EVOLUTIONARY PARAMETER  $e$  (eq. [7.8]) THAT YIELD A SLOPE OF 0.75 IN THE  $[\log N(M), m]$ -RELATION IN THE PRESENT COSMOLOGY

$z$	$\epsilon = -1$	$\epsilon = -\frac{1}{2}$
2.....	-1.70	-2.2
2.5.....	-1.68	-2.16
3.....	-1.68	-2.13
3.5.....	-1.67	-2.11

In general, eliminating  $\eta$  between (13.5) and (13.8), we obtain

$$m = m_0 + 5 \log F(z, \frac{1}{2}) + 5 \left[ \frac{\alpha - 0.6}{2\alpha - 2\alpha(1+z)^{1/2}} - 1 \right] \log(1+z), \quad (13.9)$$

where  $\alpha$  is the slope (13.5).

### b) Present Cosmology

When the same procedure is followed within the present cosmology, one gets better results. The relation (13.5) becomes, using (5.6) to eliminate  $z$  in favor of  $z$ ,

$$\text{slope} = \frac{0.6}{1 - 2(p+1) + 2(p+1)(1+z)^{(1-\epsilon)/(2+\epsilon)}}, \quad (13.10)$$

or

$\epsilon = -1$ :

$$\text{slope} = \frac{0.6}{1 + 2(p+1)(z^2 + 2z)}, \quad 8p = 3(e-1); \quad (13.11)$$

$\epsilon = -\frac{1}{2}$ :

$$\text{slope} = \frac{0.6}{1 + 2(p+1)z}, \quad 4p = e - 2. \quad (13.12)$$

Analogously, for  $\epsilon = -1$  and  $\epsilon = -\frac{1}{2}$ , equation (13.1) is given by equations (7.16) and (7.17).

Let us now require again that the slope be 0.75. We obtain for  $e$  the results shown in Table 11. Using these values in (7.16) and (7.17), we obtain the results shown in Table 12. A considerable improvement exists over the corresponding values of  $m - m_0$  of column (3), Table 10. In fact, we subtract more from  $m_0$ . As an example, let us take  $z = 2$ ,  $\log cz = 5.8$ . We obtain

standard cosmology:  $m = m_0 - 0.68$ ;

present cosmology:  $m = m_0 - 1.89(\epsilon = -1)$ ,  $m = m_0 - 1.35(\epsilon = -\frac{1}{2})$ .

Since  $m_0 \approx 21$ , when these values are reported in Figures 5-8, we get a better fit to the data with the present cosmology. The cases corresponding to  $\epsilon > 0$  are indistinguishable from those of the standard case. In conclusion, we can say that the method of determining the evolutionary parameter  $e$  from the slope of the  $\log N$  versus  $m$  relation for QSOs and then using it in the  $m$  versus  $z$  relation yields a better fit in the present cosmology than in standard cosmology.

The previous derivation could lead one to conclude that we have excluded density evolution in favor of luminosity evolution only. This, however, is not the case. To prove our point, let us suppose that we add a factor  $\beta^s$  to (12.12)

TABLE 12

THE  $(m, z)$ -RELATION (eqs. [7.16] and [7.17]) CORRESPONDING TO THE PRESENT COSMOLOGY WITH THE EVOLUTIONARY CORRECTIONS FROM TABLE 11

$z$	$m - m_0 (\epsilon = -1)$	$m - m_0 (\epsilon = -\frac{1}{2})$
2.....	-1.89	-1.35
2.5.....	-1.75	-0.95
3.....	-1.70	-0.83
3.5.....	-1.63	-0.73

to simulate a density evolution. The derivation of (13.5) then follows: The relation (13.2) between the parameter  $p$  in (3.5) and  $e$  is now changed to

$$p = -\frac{1}{2} \frac{\epsilon}{1-\epsilon} \left( \frac{3}{2}e - \frac{3}{2}g + s \right). \quad (13.13)$$

The parameter  $s$  affects  $p$  in the same way as  $3/2e$  does, and, just as in standard cosmology, one can interpret the rules in terms of luminosity or density evolution or as a combination of the two.

#### XIV. RESULTS

##### a) The $(m, z)$ -Relation

i)  $\epsilon < 0$

The cases  $\epsilon = -1$  and  $\epsilon = -\frac{1}{2}$  are presented in Figures 5 and 6 (*dashed lines*). The solid line corresponds to standard cosmology with  $\bar{q}_0 = 1$  and no evolutionary effects. The observational points of Figures 5–8 are taken from Figure 3 of Sandage (1972a). The value of the parameter  $m_0$  of equation (7.11) is fitted to be  $m_0 = 21.03$ . Several comments are in order concerning Figures 5 and 6.

First of all, the  $k = 0$ ,  $\bar{q}_0 = 0.5$  case, corresponding to the original Dirac model, equation (5.7), has been repeatedly criticized for yielding  $q_0 = 2$ , a value considered too high to fit the  $(m, z)$ -relation.

This conclusion was based on an incorrect derivation of the  $(m, z)$ -relation within the context of this new cosmology. We now see that since the  $(m, z)$ -relation depends on  $\bar{q}_0$  and not  $q_0$ , a value of  $q_0 = 2$  is perfectly acceptable since it can still correspond to an open universe.

Second, contrary to the case of standard cosmology, the  $(m, z)$ -curves are very sensitive to a change in  $\bar{q}_0$ . In fact, a change from 0.1 to 0.5 has a very distinct effect on the slope of the curves. In standard cosmology, the two curves corresponding to  $\bar{q}_0 = 0.1$  and  $\bar{q}_0 = 0.5$  are practically indistinguishable below  $m \sim 20$ .

The higher sensitivity in our case makes it possible to get a good fit even with an open universe, say with  $k = 0$ , something not possible in standard cosmology, where a bending toward lower  $m$ 's, like the one characterizing the  $\bar{q}_0 = 0.5$  curve, can be achieved only with a large  $\bar{q}_0 (> 6)$ , i.e., with a closed universe, as seen from Figure 1 of Sandage (1961a).

ii)  $\epsilon > 0$

The cases of  $\epsilon = +\frac{1}{2}$  and  $\epsilon = 0.92$  are represented in Figures 7 and 8 (*dashed lines*).

For any given  $z$ , both the  $\bar{q}_0 = 0.1$  and  $\bar{q}_0 = 0.5$  cases correspond to lower  $m$ 's than they would in standard cosmology with the same  $\bar{q}_0$ . In fact, the solid line corresponds to  $\bar{q}_0 = 1$ , i.e., a closed universe.

##### b) Angular Diameters, $\theta_m$ versus $z$

The cases  $\epsilon = -1$  and  $\epsilon = -\frac{1}{2}$  are presented in Figures 13 and 14 (*dashed lines*).

As in the  $(m, z)$ -case, we also note that within the present theory in the  $\epsilon = -1$  case a sizable difference exists between  $\bar{q}_0 = 0.1$  and  $\bar{q}_0 = 0.5$ . In standard cosmology, a much less pronounced difference exists between curves I and II.

When  $\epsilon$  is positive, the predictions of the present cosmology becomes less sensitive to the value of  $\bar{q}_0$ , to the point where no difference exists between  $\bar{q}_0 = 0.1$  and  $\bar{q}_0 = 1$ , when  $\epsilon \rightarrow 1$  (Fig. 16). In that sense the contrast between Figure 13 ( $\epsilon = -1$ ) and Figure 16 ( $\epsilon = 0.92$ ) is very instructive. An open universe with  $\bar{q}_0 = 0.1$  does provide a satisfactory fit to the data in the  $\epsilon < 0$  cases; the  $\bar{q}_0 = 0.5$  case is not as satisfactory. For positive  $\epsilon$ , no preferable  $\bar{q}_0$  exists.

The differences between the present theory and the standard one, as displayed in Figures 13–16, are mainly due to the disentangling of atomic and gravitational times. In that sense they constitute a purer cosmological probe than others, where astrophysical effects intervene very heavily.

For negative  $\epsilon$ , we conclude that values of  $\bar{q}_0$  greater than  $\frac{1}{2}$  (closed universe) would fit the data very poorly. For positive  $\epsilon$ , closed as well as open universes are acceptable.

##### c) Isophotal Angles, $\theta_i$ versus $z$

The isophotal angles  $\theta_i$  versus  $z$  given by equation (10.6) are presented in Figures 17–20 (*dashed lines*). The observational points as well as the full lines are taken from Figure 5 of Sandage (1972b). The number labeling each curve is the value of  $\bar{q}_0$ . The value  $-1$  corresponds to the steady state. Figures 17–20 are very instructive since they clearly show the sensitivity of the curves to the values of  $\bar{q}_0$ . In standard cosmology, a value of  $\bar{q}_0 = 2.5$  yields a lesser slope than does  $\bar{q}_0 = 0.5$  in the present theory. From this circumstance stems our hope of being able to discern a value of  $\bar{q}_0$ .

For negative  $\epsilon$ , an open universe fits the data certainly better than a closed one, which would predict angles too large for a given redshift. This same feature characterizes the metric diameters. The deterioration of the fit as  $\bar{q}_0$  passes  $\frac{1}{2}$  becomes less evident as  $\epsilon$  becomes positive. It is interesting that for the isophotal as well as for the metric

angles the sensitivity of the results to  $\bar{q}_0$  that characterizes negative  $\epsilon$  progressively disappears as  $\epsilon$  goes through zero ( $\beta \rightarrow 1$ , standard case) and becomes positive.

As the extreme case  $\epsilon \sim 1$ , the curves are totally insensitive to  $\bar{q}_0$ .

#### d) The $[N(m), m]$ -Relation for QSOs

As explained in the text, the present impossibility of computing theoretically the evolutionary parameter  $e$  for QSOs has been circumvented in this paper by requiring that  $e$  be such as to give a satisfactory  $[\log N(m), m]$ -relation. When such an  $e$  is used in the  $(m, z)$ -relation, we derive a curve that nicely follows the envelope of the QSO data. This again happens only if  $\epsilon < 0$ . For positive  $\epsilon$  this procedure leads to results that are no better than those of standard cosmology, where a very satisfactory fit cannot be achieved (see § XIII).

## XV. DISCUSSION

Even though it might be clear by now, we want to stress once more that the scale-covariant cosmology does not provide just one curve for the  $(m, z)$ -relation, which would make the acceptability or rejection of the theory a simple matter of comparison with the data. Such is the case for the steady state cosmology. As stated in the Introduction, our objective is analogous to that of standard cosmology, namely, the determination of the geometrical parameters that characterize the nature of spacetime. However, the central idea of our work is the recognition that *observations made with atomic instruments do not necessarily yield the geometrical parameters unless specific assumptions are made regarding the relation between atomic and gravitational dynamics*, a relation that we have embodied in the function  $\beta(t)$ . Once a particular form of the scaling function  $\beta(t)$  is chosen, the analysis of the data can be carried out exactly as in standard cosmology, employing the newly derived  $(m, z)$ -relation, equation (7.11). Given this framework, we can now assess the viability of the scale covariant cosmology.

The canonical test is given by the  $(m, z)$ -relation using elliptical galaxies. From Figures 5–8, we can appreciate the relation between the observed data and the theoretical predictions.

In the past, the  $(m, z)$ -relation was employed to discriminate against possible variations of  $G$ . For example, Tinsley and Barnothy (1973) concluded that one version of Hoyle-Narlikar (HN) theory yields an  $(m, z)$ -relation incompatible with the data. (See, however, Canuto and Narlikar 1979 for an extensive comparison of the HN theory with observations). Even though the HN theory has in common with the present one a varying  $G$ , its structure is very different, as are its physical assumptions, and no easy comparison is therefore possible.

In the present theory, for positive  $\epsilon$ , all the tests yield results very similar to the ones in standard cosmology, in the sense of being insensitive to  $\bar{q}_0$ .

For negative  $\epsilon$ , however, the fit to the data becomes certainly better since one can discriminate between an open and a closed universe. Positive  $\epsilon$  and negative  $\epsilon$  behave therefore in very distinct manners. Given the historical importance of the  $(m, z)$ -relation vis-à-vis the variation of  $G$  and the impression created by the work cited (see also Maeder 1977), we can conclude that among the many possible theoretical structures one can devise to account for a  $G(t)$ , the present one, based on a scale covariance, yields results which are either equal to or better than the ones of standard cosmology.

We have completed a description of scale-covariant cosmology and have seen how it compares with a host of observations. The fact that none of the scaling functions  $\beta(t)$  we have chosen can be ruled out by the available data is remarkable. In fact, it was not our main thrust to achieve a fit to the data superior to those attempted so far. Rather, our first aim was to show that the data are *compatible with a nonconstant  $\beta(t)$* , provided every ingredient in the computation is calculated consistently with the theory. This aim has been accomplished.

It therefore follows from the present work that there exists a *consistent alternative interpretation of the cosmological data*.

To drive home this point, we shall give the following further illustration. For a given observed  $H_0$  (see [7.11] and [7.12]), we can define a “critical density”

$$\rho_c = \frac{3H_0^2}{8\pi G}. \quad (15.1)$$

From (2.4) we have

$$\rho_0/\rho_c = 2\bar{q}_0(1 + h_0/H_0)^2. \quad (15.2)$$

Thus, from the observed average density and the value of  $\rho_c$ , as computed from  $H_0$ , we *cannot deduce directly the curvature of space*.

If  $\epsilon < 0$ ,  $h_0 > 0$ . For a given (observationally determined)  $\rho_0/\rho_c$ , the corresponding value of  $\bar{q}_0$  is smaller than for  $\epsilon = 0$ , i.e., standard cosmology. This favors an open universe. The opposite is true if  $\epsilon > 0$ .

This altogether different relation between the amount of matter in the universe and the curvature of space is perhaps the main feature of this new cosmology.

Since (15.2) depends critically on  $\epsilon$ , do we have any way to choose between  $\epsilon < 0$  and  $\epsilon > 0$ ?

As discussed elsewhere (Canuto, Hsieh, and Owen 1978), present data on time variation of the Moon's angular velocity ( $n \equiv \Omega = 2\pi/P$ ),

$$\dot{n}/n = (\dot{n}_a - \dot{n}_t)/n \quad (15.3)$$

( $\dot{n}_a$  and  $\dot{n}_t$  are the atomic and tidal contributions), seem to indicate a non-null positive result. In fact, the best data presently known for the Moon read (the units are arcsec per century<sup>2</sup>)

$$\begin{aligned} \dot{n}_t &= -30.0 \pm 3 \quad (\text{Muller 1978}); \\ \dot{n}_a &= -21.7 \pm 3.6 \quad (\text{Van Flandern 1978}); \\ \dot{n}_a &= -24.6 \pm 1.6 \quad (\text{Calame and Mulholland 1978}); \\ \dot{n}_a &= -23.8 \pm 4 \quad (\text{Williams et al. 1978}). \end{aligned} \quad (15.4)$$

Since in the present theory  $\dot{n}/n$  is directly given by  $\beta/\beta$  (see eq. [4.21] of Paper I with  $GM\beta = \text{const.}$ , eq. [2.50] of Paper I), a positive  $\dot{n}/n$  implies

$$\beta > 0, \quad (15.5)$$

i.e.,  $\epsilon < 0$ , because of (4.1).

Should (15.4) be confirmed by future analysis, it would then follow that the bulk of the cosmological tests presented in this paper favor an open universe.

Finally, a comment is in order concerning the radiation-dominated universe, characterized by the occurrence of nucleosynthesis. With the present knowledge of scale-covariant cosmology, we are in no position to make any sensible statement. Throughout our work we have learned that even the most elementary relations can be changed by the presence of  $\beta(t)$  (see, for example, eq. [2.15] in Paper II).

The study of nucleosynthesis necessitates ingredients like Fermi-Dirac distributions, invariant phase space, reaction rates, etc., that must be rederived in a manner consistent with the premises of the theory. This problem is now under study.

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## APPENDIX A EVOLUTIONARY CORRECTIONS

### A. STELLAR EVOLUTION

We shall first study the contribution to the parameter  $e$ , due to stellar evolution. Work by Maeder (1977) has already been published which treats the case when  $G$  and  $M$  are functions of time. Maeder has analyzed the two gauges (1)  $M \sim t^2$ ,  $G \sim t^{-1}$  and (2)  $M \sim t^0$ ,  $G \sim t^{-1}$ . We shall generalize his treatment so as to have results valid for an arbitrary gauge. To that end, let us call  $t$  the epoch at which a given photon leaves a galaxy and  $t_1$  the time at which the galaxy is born. All times are computed from  $t = 0$ . The so-called one-burst model for formation of elliptical galaxies is here adopted, as in all previous computations of this kind.

The contribution of dwarf stars to the total luminosity is written as

$$L_d = \int_{M_L(t)}^{M_f(t)} \frac{dN(t)}{dM(t)} l_{M(t)} dM(t), \quad (A1)$$

where  $M_f(t)$  is the turnoff mass at time  $t$ ,  $M_L(t)$  is the lower mass limit at time  $t$ ,  $dN/dM$  is the mass spectrum, and  $l_{M(t)}$  the luminosity of a star of mass  $M(t)$ . We shall write in general

$$l_{M(t)} \propto M^\alpha(t) G^\delta(t) \beta^\gamma, \quad (A2)$$

where the coefficient  $\alpha$ ,  $\delta$ , and  $\gamma$  depend on the type of opacity and nuclear reactions employed. The Salpeter mass function at time  $t = t_0$ , i.e.,

$$\frac{dN}{dM} \propto M^{-(1+\alpha)}, \quad (A3)$$

must be generalized at  $M = M(t)$ . Using the general relation (see I, eq. [2.50])

$$G(\beta)M\beta = \text{const.}, \quad (A4)$$

we obtain

$$M(t) \propto \beta^{-1}(t)G^{-1}(\beta), \quad (\text{A5})$$

so that (A3) gets generalized to

$$\frac{dN(t)}{dM(t)} \propto M^{-(1+x)}(t)[\beta(t)G(\beta)]^{-x}. \quad (\text{A6})$$

In general the main-sequence lifetime  $\tau$  scales like

$$\tau \propto M(t)/l(t) \propto M^{1-\alpha}(t)G^{-\delta}(\beta)\beta^{-\gamma}. \quad (\text{A7})$$

Therefore,

$$M(t) \propto \tau^{1/(1-\alpha)}G^{\delta/(1-\alpha)}(\beta)\beta^{\gamma/(1-\alpha)}. \quad (\text{A8})$$

For a particular mass  $M_f(t)$ , one has  $\tau = t - t_1$ , so

$$M_f(t) \propto t^{1/(1-\alpha)}G^{\delta/(1-\alpha)}(1 - t_1/t)^{1/(1-\alpha)}\beta^{\gamma/(1-\alpha)}. \quad (\text{A9})$$

Maeder, like all the other authors, has neglected the  $t_1/t$  factor. Inserting (A6) and (A2) into (A1) and integrating, one gets

$$L_d \propto \beta^{\gamma-x}G^{\delta-x}(\beta)[M^{\alpha-x}(t)]_{M_f(t)}^{M_f(t)}. \quad (\text{A10})$$

Since the main contribution comes from the upper limit, we finally have

$$L_d \propto t^{-(\alpha-x)/(\alpha-1)}G^{\delta(1-x)/(1-\alpha)-x}\beta^{\gamma(1-x)/(1-\alpha)-x}. \quad (\text{A11})$$

The equivalent expression for ordinary cosmology, with  $G = \text{const.}$  and  $\beta = 1$ , can be easily retrieved: the result coincides with that derived by Tinsley (1976).

Using now equations (4.1) and (7.9), we can rewrite (A11) in the form (7.8), i.e.,

$$L_d(t) \propto \beta^{e_d(t)}. \quad (\text{A12})$$

We have then

$$e_d = (g - 1)x + \frac{1}{\epsilon} \frac{\alpha - x}{\alpha - 1} (1 + \epsilon g \delta) - g \delta + \gamma \frac{1 - x}{1 - \alpha}. \quad (\text{A13})$$

For the commonly used value  $x = 1$ ,  $e_d$  becomes independent of  $\alpha$ ,  $\gamma$ , and  $\delta$ , i.e.,

$$e_d = g - 1 + 1/\epsilon. \quad (\text{A14})$$

Since we further assume that  $G \propto 1/t$ ,  $g\epsilon = -1$ ,  $e_d$  then becomes

$$e_d = -1. \quad (\text{A15})$$

Following other authors, we also include the effect due to red giants. Here too, we follow and generalize Maeder's results. Let  $l_g$  and  $\tau_g$  be the mean luminosity and mean lifetime in the red giant phase. The contribution to  $L$  can be written as

$$L_g = \tau_g l_g \left| \frac{dN(t)}{dM(t)} \right| \left| \frac{dM_f(t)}{dt} \right|. \quad (\text{A16})$$

Using (A6) and (A9), we then have

$$L_g \propto \tau_g l_g t^{-1-x/(1-\alpha)}\beta^{-x-\gamma[x/(1-\alpha)]}G^{-x-\delta[x/(1-\alpha)]} \left[ 1 + \delta \frac{G}{\dot{G}} t + \gamma \frac{\dot{\beta}}{\beta} t \right]; \quad (\text{A17})$$

or, using (7.9),

$$L_g \propto \tau_g l_g t^{-1-x/(1-\alpha)}\beta^{-x-\gamma[x/(1-\alpha)]}\beta^{gx+\delta g[x/(1-\alpha)]} \left[ 1 + (\gamma - \delta g) \frac{\dot{\beta}}{\beta} t \right]. \quad (\text{A18})$$

In order to recover Tinsley's expression, one has to posit  $G = \text{const.}$  and  $\beta = 1$  in either (A17) or (A18). From here on, we shall neglect the second term in parentheses with respect to unity. Transforming (A18) into an expression containing only  $\beta$ , we have ( $\tau_g l_g \propto \beta^{-1}G^{-1} \propto \beta^{g-1}$ ; see [A7])

$$L_g \propto \beta^{e_g}, \quad (\text{A19})$$

where

$$e_g = (g - 1)(x + 1) + \frac{1}{\epsilon} + \frac{x}{\epsilon(1 - \alpha)} (1 + \epsilon g \delta) - \frac{x\gamma}{1 - \alpha}. \quad (\text{A20})$$

For the total luminosity, we finally write

$$\frac{d \ln L}{d \ln \beta} = e, \quad (\text{A21})$$

where

$$\begin{aligned} e &\equiv e_a + \left( \frac{R}{R + 1} \right) e_1, \\ e_1 &= g - 1 + g\delta + \frac{1}{\epsilon} \left[ 1 + (1 + \epsilon g \delta) \frac{\alpha}{1 - \alpha} \right] + \frac{\gamma}{\alpha - 1}. \end{aligned} \quad (\text{A22})$$

For  $\epsilon g = -1$ , (A22) reduces further to

$$e_1 = -1 - \frac{1}{\epsilon} \frac{\alpha - \delta}{\alpha - 1} + \frac{\gamma}{\alpha - 1}. \quad (\text{A23})$$

The important remark to be made concerning (A23) is that the last positive term largely cancels the first one, leaving only the small contribution of the second term,  $|\alpha - \delta| < \alpha - 1$ .

## B. DYNAMICAL FRICTION

A different type of evolution that can affect the luminosity is represented by dynamical friction. The phenomenon, first described by Chandrasekhar (1943) in the context of stellar dynamics, has been recently applied to globular clusters and formation of galactic nuclei by Tremaine, Ostriker, and Spitzer (1975). Ostriker and Tremaine (1975) have pointed out that the same mechanism may be operating in clusters of galaxies, where a small galaxy is dragged into the nucleus by the collective action of the field galaxies and thereby disrupted. The computation of the increase in mass and therefore luminosity of the giant galaxy has been performed by several people besides the references quoted above (see, for example, Lecar 1974). The result is

$$M_E(t_E) \propto G_E^{-1/2} m_E^{1/2} t_E^{1/2}, \quad (\text{A24})$$

where we have included subindices  $E$  to indicate that the quantities entering this expression are to be understood in Einstein units. In fact, the equations used to arrive at (A24) involve only gravitational quantities and no atomic constants; here  $m_E$  is the mass of the galaxy eventually to be disrupted by tidal forces. Let us generalize (A24) to an arbitrary gauge. We first recall that (I, eq. [2.50]) for any object of mass  $M$ , the following relation holds:

$$\beta_A M_A G_A = \beta_E M_E G_E. \quad (\text{A25})$$

We therefore have, using  $dt_E \equiv d\bar{t} = \beta(t)dt$ ,

$$M_A \propto \frac{\beta_E G_E}{\beta_A G_A} \frac{(M_A \beta_A G_A)^{1/2}}{(\beta_E G_E)^{1/2}} \left[ \int_{t_c}^t \beta(t) dt \right]^{1/2} / G_E^{1/2}, \quad (\text{A26})$$

or finally ( $\beta_E = 1$ )

$$M \propto \frac{1}{\beta G(\beta)} \left( \int_{t_c}^t \beta(t) dt \right)^{1/2}, \quad (\text{A27})$$

where we have suppressed the index  $A$ ; the time  $t_c$  can be taken to be zero (as originally done by Ostriker and Tremaine) or more realistically as a given fraction of the Hubble time. In fact galaxies formed not at  $t = 0$ , but after decoupling. We shall leave  $t_c$  undetermined for the time being.

Equation (A27) is the generalization of equation (8) of Tremaine *et al.* to our cosmological framework. We must note that (A27) is exact to the same extent to which Tremaine *et al.* is. In fact, our generalization has not introduced any approximations. We can now write, for the contribution to  $L$  by dynamical friction, the expression

$$L_{\text{dy. fr.}} \propto \beta^{g-1} \left[ \int_{t_c}^t \beta(t) dt \right]^{1/2}. \quad (\text{A28})$$

For small look-back times, we can approximate the bracket so that

$$L_{\text{dy. fr.}} \propto \beta^{g-1}(t) \left( 1 - \frac{1}{2} \frac{z}{H_0 t_0} \frac{1}{1 - t_c/t_0} \right). \quad (\text{A29})$$

Using now (5.13), we finally get

$$L_{\text{dy. fr.}} \propto \beta^{g-1}(t) \beta^{-1/[2\epsilon(1-t_0/t_0)]}. \quad (\text{A30})$$

## APPENDIX B

### THE $m-z$ RELATION

We shall now present an alternative derivation of the magnitude versus redshift relation using a Robertson-Walker metric:

$$ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right].$$

The tangent vector of a photon path, extending from the origin radially outward, can be written as (see II, eq. [2.26])

$$k^\mu = \frac{\nu_0}{\beta R} \left[ 1, \frac{(1 - kr^2)^{1/2}}{R}, 0, 0 \right], \quad (\text{B1})$$

where  $\nu_0$  is a constant. We recall that (B1) satisfies the null in-geodesic equation. The energy-momentum tensor of radiation emerging from the source, located at the origin, can be represented by

$$T^{\mu\nu} = E k^\mu k^\nu. \quad (\text{B2})$$

The modified energy momentum conservation equation and the null in-geodesic equation imply (see eqs. [2.36], [2.37] of II)

$$\begin{aligned} (Ek^\mu)_{*\mu} &= 0 = (Ek^\mu)_{;\mu} + (1 - \pi_g)Ek^\mu(\ln \beta)_{,\mu} \\ &= \frac{1}{g^{1/2}}(g^{1/2}Ek^\mu)_{,\mu} + (1 - \pi_g)Ek^\mu(\ln \beta)_{,\mu}, \end{aligned}$$

where  $g^{1/2} = [-\det(g_{\mu\nu})]^{1/2}$ . Making use of (B1), the above equation can be rewritten as

$$\frac{\nu_0}{R^3} \left( \frac{ER^2}{\beta} \right)_{,t} + \frac{\nu_0}{R^2} \frac{(1 - kr^2)^{1/2}}{r^2} \left( \frac{Er^2}{\beta} \right)_{,r} + (1 - \pi_g)E \left[ \frac{\nu_0}{\beta R} \frac{\beta_{,t}}{\beta} + \frac{\nu_0}{\beta R^2} (1 - kr^2)^{1/2} \frac{\beta_{,r}}{\beta} \right] = 0.$$

Next, we introduce the variables  $\tau$  and  $\xi$ :

$$d\tau = \frac{dt}{R(t)}; \quad d\xi = \frac{dr}{(1 - kr^2)^{1/2}},$$

so that the above equation becomes

$$\frac{\nu_0}{\beta^{1-\pi_g} R^4 r^2} [\mathcal{E}_{,\tau} + \mathcal{E}_{,\xi}] = 0,$$

where

$$\mathcal{E} \equiv ER^2 r^2 \beta^{-\pi_g}.$$

Hence

$$\mathcal{E} = \mathcal{E}(\tau - \xi). \quad (\text{B3})$$

Since on a null path  $\tau - \xi$  is constant,  $\mathcal{E}$  is a conserved quantity along the beam of radiation. Next we note that the observed flux, which is equivalent to the apparent luminosity, is given by

$$l = c\rho_\gamma = cT^{\mu\nu}u_\mu u_\nu = cE(u_\mu k^\mu)^2 = cE(\nu^2/\beta^2),$$

where equation (6.5) of II has been used. Thus, we can write equation (B3) as

$$\frac{l_1}{c} \frac{\beta_1^{1-\pi_g}}{\nu_1^2} r_1^2 R_1^2 = \frac{l_2}{c} \frac{\beta_2^{1-\pi_g}}{\nu_2^2} r_2^2 R_2^2.$$

Subscripts 1 and 2 denote different space time points along the beam of radiation. As we approach the source,

$$\lim_{r \rightarrow 0} 4\pi r^2 R^2 l = L,$$

where  $L$  is the intrinsic luminosity. Hence,

$$\begin{aligned} l_{\text{obs}} &= \frac{L}{4\pi} \left( \frac{\beta_{\text{em}}}{\beta_{\text{obs}}} \right)^{2-\pi_g} \left( \frac{v_{\text{obs}}}{v_{\text{em}}} \right)^2 \frac{1}{r_{\text{obs}}^2 R_{\text{obs}}^2}, \\ l_{\text{obs}} &= \frac{L}{4\pi r_{\text{obs}}^2 R_{\text{obs}}^2} \frac{1}{(1+z)^2} \left( \frac{\beta_{\text{em}}}{\beta_{\text{obs}}} \right)^{2-\pi_g}, \end{aligned} \quad (\text{B4})$$

which coincides with (7.4a).

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